Introduction to Economic and Social Network

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Briefly summarizes the notes I take for IESN, 25 Fall, Peking University.

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Thanks to typst!

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1. Games on Network

1.1. Games on Network

Start with a canonical case:

- Each player chooses $x_i = 0, 1$
- Payoff depends on
  - how many neighbors choose each action
  - how many neighbors a player has

Consider cases where i's payoff is

$$u_{d_i}\left(x_i, m_{N_i}\right) \tag{1}$$

where m_{N_i} is the number of neighbors of i choosing 1 and $d_i(g)$ is the number of neighbors of i.

Or we should let N_i be the set of i's neighbors, then $d_i(g) = |N_i|$ and

$$m_{N_i} = \#\left\{j \in N_i, x_j = 1\right\} \tag{2}$$

Example 1.1.1 (Simple Complement): Agent i is willing to choose 1 $\Longleftrightarrow$ at least t neighbors do:

- payoff action 0: $u_{d_i}\left(0, m_{N_i}\right) = 0$
- payoff action 1: $u_{d_i}\left(1, m_{N_i}\right) = -t + m_{N_i}$

We try to analyze the game in Example 1.1.1, the difference is

$$u_{d_i}\left(1, m_{N_i}\right) - u_{d_i}\left(0, m_{N_i}\right) = -t + m_{N_i} \tag{3}$$

Then the best response is:

$$x_i^*(m) = \begin{cases} 1 \text{ if } m \geq t \\ 0 \text{ o.w.} \end{cases} \tag{4}$$

Now let $t = 2$:

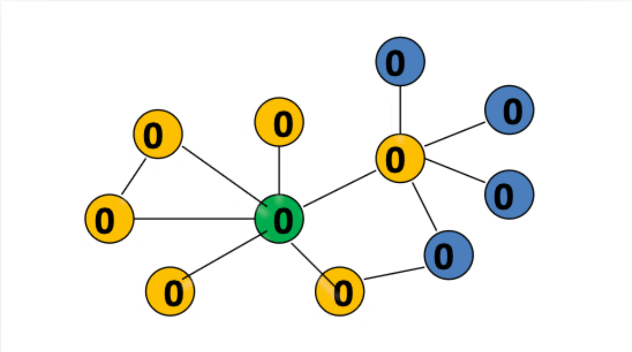


Figure 1: eq.1 – all choose 0

An agent is willing to take action 1 if and only if at least two neighbors do. And another equilibrium is as in Figure 2:

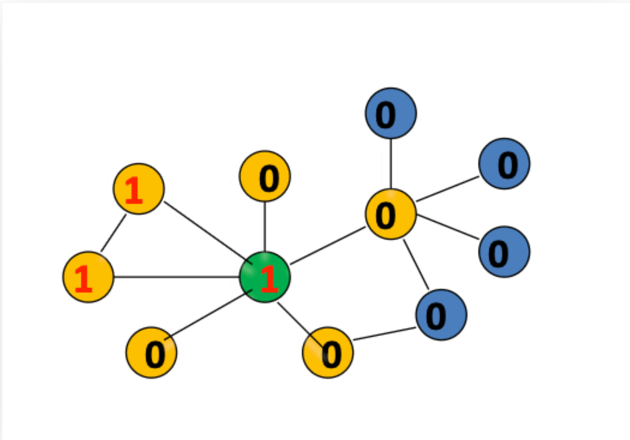


Figure 2: eq.2 – 3 nodes chooses 1

1.2. Complements and Substitutes

Definition 1.2.1 (Externalities) :
- Others' behaviors affect my utility/welfare
- Others' behaviors affect my decisions, actions, consumptions, opinions...

Key to analysis: others' actions affect the relative payoffs to my behaviors

1.2.1. Complements

Choice to take an action by my friends increases my relative payoff to taking that action (e.g., friend learns to play a video game)

Let $m = m_{N_i}$ as defined in Equation 2, then strategic complements means:

Definition 1.2.2 (Complements): for all d[1] and $m \geq m'$, we have

$$u_d(1, m) - u_d(0, m) \geq u_d(1, m') - u_d(0, m') \tag{5}$$

So this basically means that if others choose 1 more often, then I'm more (at least weak) likely to choose 1.

1.2.2. Substitutes

Choice to take an action by my friends decreases my relative payoff to taking that action (e.g., roommate buys a stereo/fridge)

Similarly, this can be viewed as

Definition 1.2.3 (Substitutes): for all $d \wedge m \geq m'$:

$$u_d(1, m) - u_d(0, m) \leq u_d(1, m') - u_d(0, m') \tag{6}$$

Definition 1.2.4 (Nash Eq.): Every player's action is optimal for that player given the actions of others

We often look for pure strategy equilibria and may require some mixing to get existence (hopefully not...)

1.3. Strategic Complements

Example 1.3.1 (Examples):
- education decisions
- care about number of neighbors, access to jobs, etc. – invest if at least k neighbors do
- smoking & other behavior among teens, peers, ...
- technology adoption – how many others are compatible...
- learn a language, ...
- cheating, doping

One useful observation is that complements $\Longrightarrow$ there exists a threshold $t(d)$ s.t. i prefers

[1] Note that we're talking about agent i so we basically simplify the notation by removing i out of the subscript.

$$\text{BR}_i = \begin{cases} 1 \text{ if } m_{N_i} > t(d) \\ 0 \text{ if } m_{N_i} < t(d) \end{cases} \tag{7}$$

and can be indifferent at the threshold $t(d)$.

Consider a case where t doesn't depend on d: $t \equiv 2$, then there are multiple equilibria as in Figure 3.

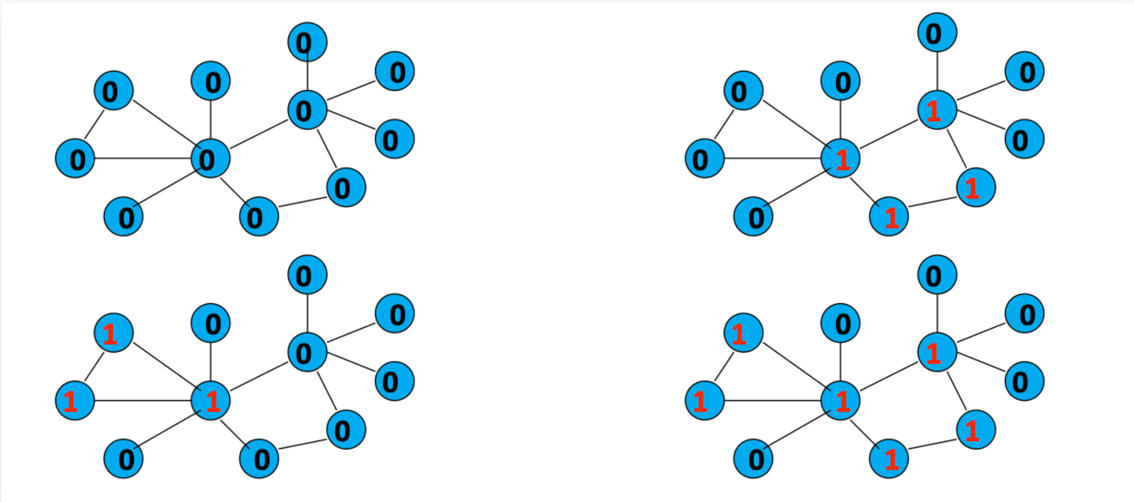


Figure 3: Multiple Equilibria

1.3.1. Lattice Structure

Moreover, one important feature is that the eq. in Figure 3 has so called *lattice structure*.

> **Definition 1.3.1** (Complete Lattice): For a set of equilibria $\mathbb{X}$,
> - there exists an eq. x' s.t. $x' \geq x, \forall x \in \mathbb{X}$
>
> and
> - there exists an eq. x'' s.t. $x'' \leq x, \forall x \in \mathbb{X}$

But first, we need to define the operators used:

> **Definition 1.3.2** (Coordinate order): For 2 action allocation: $x = (x_i), y = (y_i)$, we define $x \leq y$ as
> $$x \leq y \iff x_i \leq y_i, \forall i = 1, 2, ..., n \tag{8}$$

Then given Definition 1.3.2, we can express the Definition 1.3.1 as the following assertions:

1. The maximum value at any given number of equilibrium coordinates is still in equilibrium.

2. The minimum value at any point of any number of equilibrium coordinates is still in equilibrium.
3. In particular, there exists a unique minimum and maximum equilibrium.

The following propositions exists:

Proposition 1.3.1: Given the game of strategic complements where the individual strategy sets are complete lattices, we have, *the set of pure strategy eq. are a nonempty complete lattice.*

And Proposition 1.3.1 gives the following implications:
1. eq. always exists
2. well-defined maximum and minimum eq.
3. easy to fine such eq.

1.3.2. Deeper into Proposition 1.3.1

Note that one important thing in Proposition 1.3.1 is to ensure **the individual strategy sets are complete lattices**.

Assign a **coordinate-wise/product order** to each player's strategy space. Under this order, the space has a minimum upper bound (join) and a maximum lower bound (meet) for any subset and remains within the space. Equivalently, it contains a global minimum and a global maximum, and is closed with respect to the coordinates at `max/min`.

Formally, let $S_i \subset \mathbb{R}^{k_i}$ be the action space of agent i, and the product order is defined as in Definition 1.3.2. Then S_i is a complete lattice $\iff \forall A \subseteq S_i$, we have

$$\bigvee A \in S_i, \bigwedge A \in S_i \tag{9}$$

that is to take sup or inf at each dimension. Then we know that there are

$$\perp_i \in S_i, \top_i \in S_i \tag{10}$$

which are the minimum and the maximum of S_i.[2]

1.4. An Application in Labor

Dropout decisions as in (Calvó-Armengol & Jackson, 2004; 2007). Value to being in the labor market depends on # of friends in the market
- Dropout if some # of friends do so
- Some heterogeneity in threshold (different costs, natural abilities…)
- Homophily – segregation in network

[2] And therefore usually $\top_i > \bigvee A$ unless $A = S_i$.

Consider this cascade model as in Figure 4, first, 2 groups exhibit homophily, and secondly, *one agent dropout if at least half of its neighbors do so*.

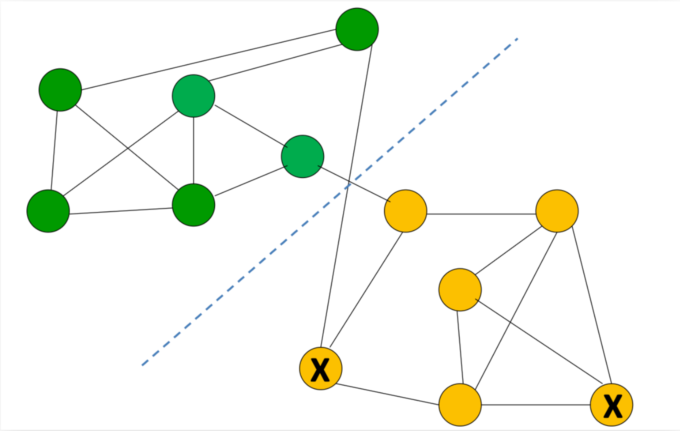


Figure 4: Cascade model

As in Figure 4, there are 2 initial dropouts. The end is that finally it reaches the Figure 5 where the whole yellow component drop out.

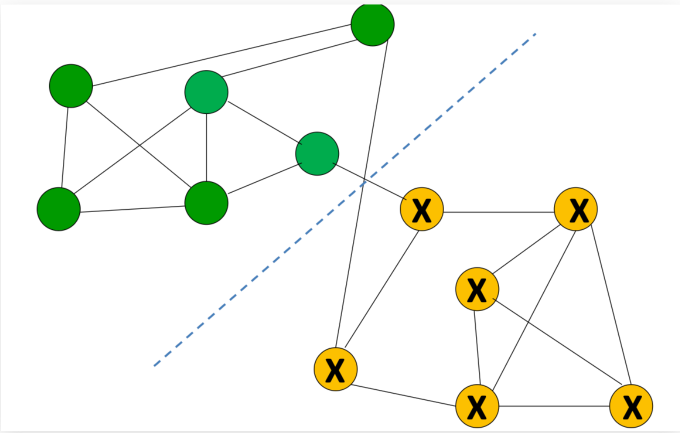


Figure 5: Cascade model: result

That is *persistent difference across groups*

1.5. Strategic Substitutes

Similarly, consider the following Example 1.5.1:

Example 1.5.1:
- information gathering
  - e.g., payoff of 1 if anyone in neighborhood is informed, cost to being informed ($c < 1$)
- local public goods (shareable products…)
- competing firms (oligopoly with local markets)

And similar to Equation 7, we have the threshold function $t(d)$ s.t.

$$\mathrm{BR}_i = \begin{cases} 1 \text{ if } m_{N_i} < t(d) \\ 0 \text{ if } \ m_{N_i} > t(d) \end{cases} \tag{11}$$

and can be indifferent at $m_{N_i} = t(d)$.

Example 1.5.2 (Best Shot Game): One is willing to choose 1 $\iff$ no neighbors do. Each node i chooses action $x_i \in \{0, 1\}$ where $x_i = 1$ means providing a public good at cost c. The payoff is

$$u_{d_i}(0, m_{N_i}) = \begin{cases} 1 \text{ if } m_{N_i} > 0 \\ 0 \text{ o.w.} \end{cases}, u_{d_i}(1, m_{N_i}) = 1 - c \tag{12}$$

and let $c \in (0, 1)$.

Then for Example 1.5.2, suppose a network in form of Figure 6,

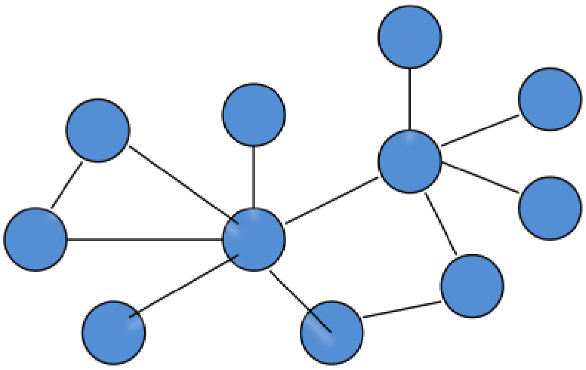


Figure 6: Network structure for Example 1.5.2

It's clear that one will choose $x_i = 1$ if and only if $m_{N_i} = 0$. One eq. is described in Figure 7.

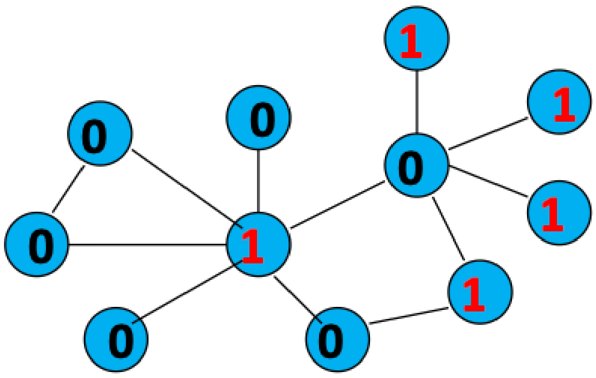


Figure 7: An eq. of Example 1.5.2

Do we have any other equilibria?

Consider the best shot game in Example 1.5.2 for a star network, then we have only 2 eq. as in Figure 8. Note that in this case there are *different distribution of utilities and different total costs.* The eq. is related to a concept called **maximal independent set**.

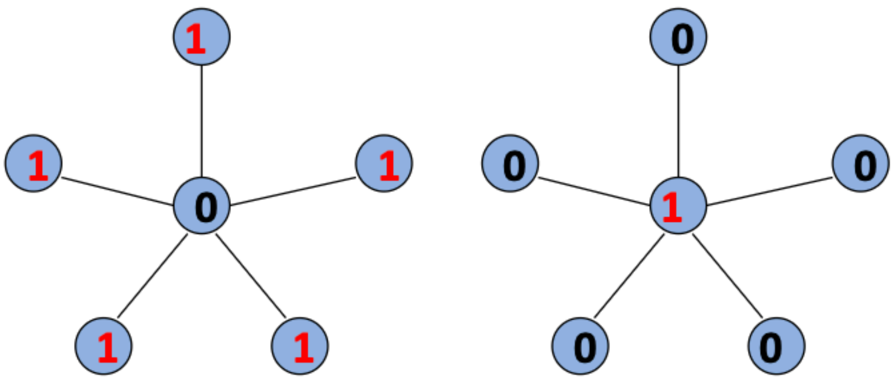


Figure 8: eq. of star network

Definition 1.5.1 (Maximal Independent Set): Let $M \subseteq N$ be the set of providers:

$$M \equiv \{i \in N, x_i = 1\} \tag{13}$$

1. independent: for all $i, j \in M$, ij $\notin g$
2. maximal: for all $k \notin M$, if we add k to $M \cup \{k\}$, then $M \cup \{k\}$ is not *independent*.

1.6. Summary

In a game of strategic **complements**:
1. Pure strategy equilibria are a nonempty complete lattice
2. A series of good properties

In a game of strategic **substitutes**:

1. Best shot game: pure strategy equilibria exist and are related to maximal independent sets
2. Others: pure strategy may not exist, but mixed will (with finite action spaces)
3. Equilibria usually do not form a lattice

2. TA session 1

2.1. Nodes and Network

Definition 2.1.1 (Nodes): set

$$N = \{1, 2, ..., n\} \tag{14}$$

Definition 2.1.2 (Network): A network (N, g) consists of nodes N and an $n \times n$ adjacency matrix g where:
- $g_{ij} = 1 \Longrightarrow i$ and j are linked
- A network is undirected iff $g_{ij} = g_{ji}$
- Usually $g_{ii} = 0$

g can also be a collection of links, say, $g = \{12, 23\}$, then $(ij) \in g$ means i and j are linked.

2.2. Neighborhood and degree

Definition 2.2.1 (Neighborhood of i): The set

$$N_i(g) \equiv \left\{j \in N : g_{ij} = 1\right\} \tag{15}$$

similarly, degree is then defined as

Definition 2.2.2 (Degree):

$$d_i(g) = \# N_i(g) \tag{16}$$

2.3. Connecting Nodes

There are some important concepts too.

Definition 2.3.1 (Walk): A walk from i to j is a sequence of nodes $\{i_t\}_{t=1}^{K}$ and sequence of links $\{i_t i_{t+1}\}_{t=1}^{K-1}$ such that

$$i_t i_{t+1} \in g, \forall k \in \{1, 2, ..., K-1\} \tag{17}$$

and $i_1 = i, i_K = j$.

Definition 2.3.2 (Path): A path is a walk with each node in $\{i_t\}_{t=1}^K$ is **distinct.**

Another important form of walk is cycle:

Definition 2.3.3 (Cycle): A cycle is a walk $\{i_t\}_{t=1}^K$ w/
1. $i_1 = i_K$
2. all other nodes are distinct

Finally we have a harder concept:

Definition 2.3.4 (Geodesic): The shortest path between 2 nodes.[3]

2.4. Component

We first start with the concept of connection.

Definition 2.4.1 (Connected): A network (N, g) is connected if there is a **Path** between every 2 nodes.

Then we can try to define component:

Definition 2.4.2 (Component): A non-empty subnetwork (N', g') such that:$N' \subseteq N, g' \subseteq g, N' \neq \emptyset$ and
- (N', g') is connected
- if $i \in N'$ and $ij \in g$, then $j \in N'$ and $ij \in g'$

And the components of a network are the maximal connected subgraphs.

Definition 2.4.3 (Complete network): All possible links are present:

$$g = \mathbf{1}\mathbf{1}^\top - I \tag{18}$$

or equivalently, $g_{ij} = 1$ for all $i \neq j$.

[3]If there is no path between 2 nodes, then the distance between them is ∞.

2.5. Diameter and average path length

We begin with the definition of diameter:

Definition 2.5.1 (Diameter): The largest geodesic as defined in Definition 2.3.4 between any 2 nodes in the network.

Actually, since there are some special cases as in footnote[3], the diameter is rigorously ∞. However, in practice we usually then look at the diameter of giant component instead.

For a tree with n nodes, where each node has a branching factor of d links, the diameter (D) is proportional to the logarithm of the number of nodes to the base of the branching factor.

$$D \propto \frac{\log(n)}{\log(d)} \tag{19}$$

Then there comes the average path length:

Definition 2.5.2 (Average path length): The average geodesic between any 2 nodes in the network, or formally,

$$\sum_{i \geq j} \ell(i,j) \left[\frac{n(n-1)}{2} \right]^{-1} \tag{20}$$

2.6. Centrality Measures

This basically measures the importance of a node. Firstly we have the degree centrality, which measures *how connected a node is*.

Definition 2.6.1 (Degree Centrality): Let $d_i(g)$ be the degree of node i where

$$d_i(g) = \#\left\{ j \in N, g_{ij} = 1 \right\} \tag{21}$$

or in the normalized form:

$$\frac{1}{n-1} d_i(g) \tag{22}$$

Then we have the closeness one:

Definition 2.6.2 (Closeness Centrality): Let

$$C_i(g) = \frac{n-1}{\sum_{j=1}^{n} l(i,j)} \tag{23}$$

where $l(i,j)$ is the number of links in the shortest path between i and j.

For decay centrality, we have

Definition 2.6.3 (Decay Centrality): Let

$$C_i^d(g) = \sum_{j=1, j\neq i}^{n} \sigma^{l(i,j)} \tag{24}$$

where $l(i,j)$ is defined similar to that of Definition 2.6.2. Its normalized form is

$$\frac{1}{(n-1)\sigma} C_i^d(g) \tag{25}$$

- $\sigma \to 1$, decay centrality measures component size.
  - to see this, note that

$$\begin{aligned}\lim_{\sigma\to 1} C_i^d(g) &= \lim_{\sigma\to 1} \sum_{j=1, j\neq i}^{n} \sigma^{l(i,j)} \\ &= \lim_{\sigma\to 1} \sum_{j=1, j\neq i}^{n} 1^{l(i,j)}\end{aligned} \tag{26}$$

  which is just the size of the component that i is in![4]
- $\sigma \to 0$, decay centrality measures degree .(gives infinitely more weight to closer nodes than farther nodes)
  - That's because

$$\lim_{\sigma\to 0} \frac{\sigma^1}{(n-1)\sigma} = \frac{1}{n-1}, \lim_{\sigma\to 0} \frac{\sigma^l}{(n-1)\sigma} = 0, \forall l \geq 2, l \in \mathbb{N} \tag{27}$$

- σ in between, decaying distance measure.(very distant nodes are weighted less than closer nodes)

Now we move to Betweenness centrality, which measures *how important a node is in terms of connecting other nodes.*

[4] However, this is certainly not $d_i(g)$.

Definition 2.6.4 (Betweenness Centrality): Let

$$C_k^B(g) = \frac{\sum_{i<j, i\neq k} \frac{P_k(ij)}{P(ij)}}{\binom{n-1}{2}} = \frac{2\sum_{i<j, i\neq k} \frac{P_k(ij)}{P(ij)}}{(n-1)(n-2)} \tag{28}$$

be the betweenness centrality of node k[5], where
1. $P_k(ij)$ denotes the total number of geodesics between i, j that k lies on.
2. $P(ij)$ is the total number of geodesics between i, j

2.7. Prestige, Influence, Eigenvector-based Centrality

Now we are trying to find sth really hard. This centrality considers not only how many neighbors a node has (i.e., degree centrality), but also how important those neighbors are. A node that connects to many important nodes (a high-centrality node) will also have high centrality.

Definition 2.7.1 (Eigenvector Centrality): Given a network (N, g), we define the eigenvector centrality of a node i as:

$$C_i^e(g) = \alpha \sum_{j\in N} g_{ij} C_j^e(g) \tag{29}$$

and in vector form:

$$C^e = \alpha g C^e \tag{30}$$

where $C^e(g) \in \mathbb{R}^n$.

Actually you may (or at least I'm) unhappy with Equation 30, so we rearrange it to $\frac{1}{\alpha}C^e = gC^e$, so let $\lambda = \frac{1}{\alpha}$, this is just the definition of eigenvalue of g. Moreover, we are looking for the eigenvector associated with the **largest** eigenvalue which is nonnegative.

Then we turn to Katz-Bonacich Centrality.

2.7.1. Intuition

The intuition is that *Each node has a base value, then receiving extra values from others, via direct or indirect connections.*

2.7.2. Number of walks from i to j

Consider a network (N, g), then g^n indicates how many walks there are of length n between 2 nodes, see Example 2.7.1.

[5] I use $\binom{n-1}{2}$ to normalize it cuz we basically want 2 nodes out of $n-1$ nodes to be connected, and thereby trying to find their geodesics.

Example 2.7.1: Let $g^2 = \begin{pmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{pmatrix}$, then $g_{14} = 2$ means that there are 2 walks between $1, 4$ of length 2.

Then we try to get each node an initial value: $\alpha d_i(g)$ for some $\alpha > 0$.

1. we add in all walks of length 1 from j to i w/

$$\sum_{j:ij\in g} \beta\alpha d_j(g) \tag{31}$$

2. then we add in all walks of length 2:

$$\sum_{j:ij\in g} \beta^2\alpha d_j(g) \tag{32}$$

3. ...
4. finally it looks like

$$\begin{aligned} C^b(g) &= \alpha g\mathbf{1} + \alpha\beta g^2\mathbf{1} + \ldots \\ &= \alpha\left(1 + \beta g + \beta^2 g^2 + \ldots\right)g\mathbf{1} \\ &= \alpha(I - \beta g)^{-1} g\mathbf{1} \end{aligned} \tag{33}$$

   and note that $g\mathbf{1} \in \mathbb{R}^{n\times 1}$ is actually the degree of each node.

- β captures how the value of being connected to someone decays with distance.
- α captures the base value on each node.

2.8. Clustering

"If i has relationships with both j and k, how likely is it that j and k are related ?"

This problem is central to clustering. In sociology, this is known as a **triadic closure**: If A knows B and A knows C, then B and C are very likely to know each other as well.

2.8.1. Individual Clustering Coefficient

Definition 2.8.1 (Individual Clustering Coefficient): Let

$$Cl_i(g) = \frac{\#\{jk \in g | j,k \in N_i(g)\}}{\#\{jk \mid j,k \in N_i(g)\}} \tag{34}$$

be the Individual Clustering Coefficient of node i.

And it's easy to see that

$$\#\{jk \mid j,k \in N_i(g)\} = \binom{d_i(g)}{2} \tag{35}$$

1. In the Erdős–Rényi (E-R) random network $G(n, p)$, the probability of a connection between any two nodes is p. *Since the links in the E-R model are formed independently, the probability of a connection between two neighbors j and k of node i is the same as the probability of any connection in the network p* and therefore

$$Cl_i(g) = p. \tag{36}$$

   This indicates that there is no clustering phenomenon exceeding the random level in the E-R model.
2. If the degree of node i is $d_i(g) \leq 1$ (i.e., it has 0 or 1 links), then it has insufficient neighbors to form any pair of neighbors. Then

$$Cl_i(g) = 0 \tag{37}$$

2.8.2. Overall clustering

Definition 2.8.2 (Overall clustering): Let

$$Cl(g) = \frac{\sum_{i \in N} \#\{jk \in g \mid j, k \in N_i(g)\}}{\sum_{i \in N} \#\{jk \mid j, k \in N_i(g)\}} \tag{38}$$

be the overall clustering coef. of a network (N, g).

Equal weight for each pair of friends; High-degree nodes get more weights.
1. High-weight nodes have a greater impact: This method is essentially counting triangles. Nodes with higher degrees (more connections) can form more triples (whether open or closed), therefore their clustering behavior contributes more to the final $Cl(g)$ value.
2. "Equal weight for each friend" means that each "triple" contributes the same amount when calculating the ratio.

2.8.3. Average clustering

Definition 2.8.3 (Average clustering): Let

$$Cl^{\text{avg}}(g) = \frac{1}{n} \sum_{i=1}^{n} Cl_i(g) \tag{39}$$

be the average clustering coefficient of a network (N, g).

Equal weight for every node.

3. Games on Networks II

3.1. Strategic Substitutes (Bramoullé & Kranton, 2007)

You need to acquire information for an application, the payoff is

$$f\left(x_i + \sum_{j\in N_i(g)} x_j\right) - cx_i \tag{40}$$

where f is a concave "production function", FOC gives

$$f'(x^*) = c \tag{41}$$

So in all pure strategy Nash Equilibria, we have the optimization:

$$\begin{aligned} \max_{x_i} \quad & f\left(x_i + \sum_{j\in N_i(g)} x_j\right) - cx_i \\ \text{s.t.} \quad & x_i \geq 0 \end{aligned} \tag{42}$$

then KKT condition of Equation 42 gives:

$$\begin{gathered} f'\left(x_i + \sum_{j\in N_i(g)} x_j\right) = c - \lambda_i \\ \lambda_i \geq 0 \\ x_i \geq 0 \\ \lambda_i x_i = 0 \end{gathered} \tag{43}$$

and that is

$$\begin{gathered} x_i + \sum_{j\in N_i(g)} x_j \geq f'^{-1}(c) \\ x_i = 0 \ \text{ if } \ x_i + \sum_{j\in N_i(g)} x_j > f'^{-1}(c) \end{gathered} \tag{44}$$

So there are basically 2 types of equilibria

- Distributed eq.

$$x_i \in \left(0, f'^{-1}(c)\right) \tag{45}$$

  for some i's
- Specialized eq. for each i, either $x_i = 0$ or $x_i = f'^{-1}(c)$

We consider the case where $x^* = 1$, that is, everyone wants a total investment of at least 1 in her neighborhood. The equilibria is described in Figure 9.

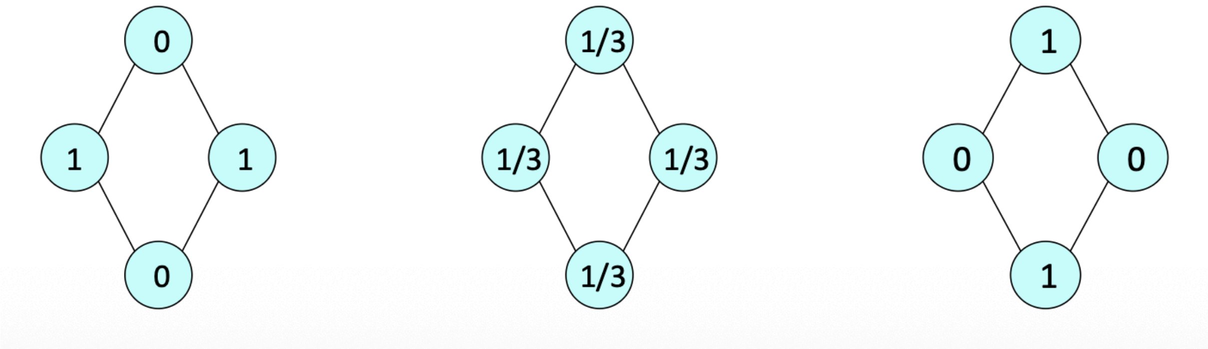


Figure 9: Eq. of BK 07

So there are more eq. but what do we know about them? Recall the definition of Definition 1.5.1, there is a new proposition about that:

Proposition 3.1.1: Specialized Nash equilibria are such that a maximal independent set = the specialist set

$$\{i : i \in N, x_i = f'^{-1}(c)\} \tag{46}$$

3.2. A linear quadratic model

This comes from (Ballester et al., 2006), we set the payoff function as:

$$u(x_i, x_{-i}) = ax_i - 0.5bx_i^2 + \sum_j w_{ij}x_ix_j \tag{47}$$

and we set quadratic intrinsic value to be $v(x_i) = ax_i - 0.5bx_i^2$. This is also Strategic Complements since

$$\frac{\partial u}{\partial x_i} = a - bx_i + \sum_j w_{ij}x_j \tag{48}$$

so when x_j increases, $\frac{\partial u}{\partial x_i}$ also increases.

Then we try to solve the best response of x_i given x_{-i}. Let FOC:

$$a - bx_i + \sum_j w_{ij}x_j = 0 \tag{49}$$

so

$$x_i^* = \frac{1}{b}\left(a + \sum_j w_{ij}x_j\right) \tag{50}$$

consider the matrix form where $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ and $\boldsymbol{\alpha} = \frac{a}{b}\mathbf{1}_n, \boldsymbol{g} = \left(g_{ij}\right)_{n \times n}$ where $g_{ij} = \frac{w_{ij}}{b}$, then we can rewrite Equation 49 in matrix form:

$$\boldsymbol{x} = \boldsymbol{\alpha} + \boldsymbol{g}\boldsymbol{x} \tag{51}$$

and then by simple algebra we have

$$\boldsymbol{x} = \sum_{k \geq 0}^{\infty} \boldsymbol{g}^k \boldsymbol{\alpha} \tag{52}$$

and if $(\boldsymbol{I} - \boldsymbol{g})$ is invertible then we have directly

$$\boldsymbol{x} = (\boldsymbol{I} - \boldsymbol{g})^{-1} \boldsymbol{\alpha} \tag{53}$$

The model is based on **Strategic Complements**:

- If my neighbors work harder ($x_j \uparrow$), my marginal benefit increases.
- I respond by working harder ($x_i \uparrow$).
- This causes my neighbors to work even harder in response.

Without a counter-force, this cycle creates a "snowball effect," potentially causing the equilibrium actions to explode to infinity ($x_i \to \infty$).

To guarantee a unique and finite Nash Equilibrium exists, the cost of action must outweigh the network benefits. This is why the slide states we need **large** b and/or **small** w_{ij}.

- w_{ij} **(The Accelerator):** Represents the strength of the network synergy. If w is too large, the feedback loop is too strong.
- b **(The Brake):** Represents the convexity of the cost function (diminishing returns).
  - A **large** b means marginal costs rise steeply, making it too expensive for agents to increase x_i indefinitely, effectively "dampening" the feedback loop.

Recall Katz - Bonacich centrality:

$$B(g) = (I - g)^{-1} g \mathbf{1} = \sum_{k \geq 0} g^{k+1} \mathbf{1} \tag{54}$$

so we basically have

$$x \propto B(g) \tag{55}$$

1. Natural feedback from complementarities, actions relate to the total feedback from various positions
2. Related to centrality: relative number of weighted influences going from one node to another

3.3. Friendship Paradox

Example 3.3.1 (Friendship Paradox) : Who is more popular? You, or your friends? That is $\frac{1}{n}\sum_{i\in N} d_i(g)$ vs. $\frac{1}{n}\sum_{i\in N}\left(\frac{1}{d_i(g)}\sum_{j\in N_i(g)} d_j(g)\right)$.

We try to solve this rigorously :-)

For the LHS, we simply denote that

$$\mu = \mathbb{E}[d_i(g)] = \frac{1}{n}\sum_{i\in N} d_i(g) \tag{56}$$

and for the RHS, we should consider the probability of a friend j to be selected. We know that in this case: $P(j) \propto d_j(g)$, and by regularity, we have $\sum_{j\in N, d_j(g)\geq 1} P(j) = 1$, so we have

$$P(j) = \frac{d_j(g)}{\sum_{k\in N, d_k(g)\geq 1} d_k} = \frac{d_j(g)}{\sum_{k\in N} d_k} \tag{57}$$

so we have the RHS as:

$$\begin{aligned}
\frac{1}{n}\sum_{i\in N}\left(\frac{1}{d_i(g)}\sum_{j\in N_i(g)} d_j(g)\right) &= \sum_{j\in N, d_j(g)\geq 1} d_j(g)P(j) \\
&= \sum_{j\in N, d_j(g)\geq 1} d_j(g)\frac{d_j(g)}{\sum_{k\in N} d_k} \\
&= \frac{1}{\sum_{k\in N} d_k}\sum_{j\in N} d_j^2(g) \\
&= \frac{1}{n\mu}\sum_{j\in N} d_j^2(g)
\end{aligned} \tag{58}$$

consider the variance, we have

$$\mathbb{V}[d_i(g)] = \mathbb{E}\left[d_i^2(g)\right] - (\mathbb{E}[d_i(g)])^2 \tag{59}$$

so

$$\begin{aligned}
\mathbb{E}\left[d_i^2(g)\right] &= \mathbb{V}[d_i(g)] + (\mathbb{E}[d_i(g)])^2 \\
&= \sigma^2 + \mu^2
\end{aligned} \tag{60}$$

and

$$\text{RHS} = \frac{1}{\mu}\mathbb{E}\left[d_i^2(g)\right] = \frac{1}{\mu}(\sigma^2 + \mu^2) = \frac{\sigma^2}{\mu} + \mu > \mu = \text{LHS} \tag{61}$$

and that's basically the proof.

Example 3.3.2 (The Impact of the Friendship Paradox): Consider a 0-1 game...
- People choose solid or plaid
- Prefer to match the majority of your friends
- Go with own preference if indifferent

People typically have fewer friends than their friends
1. People with more friends are 'seen' more
2. Popular people are over-represented in the views of others

So basically we start with this state:

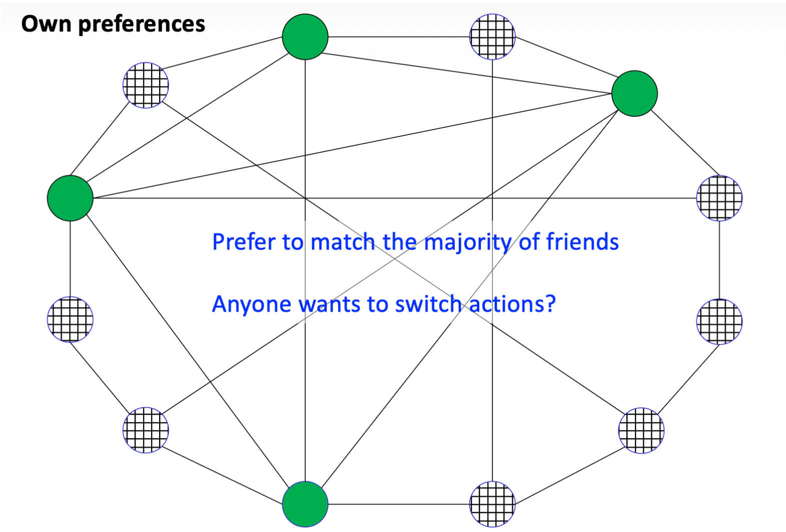


Figure 10: Own preferences

Note that in this starting state, only the influencers prefer the *solid* ones. Then each node tries to match the majority of its friends. Finally it becomes like this:

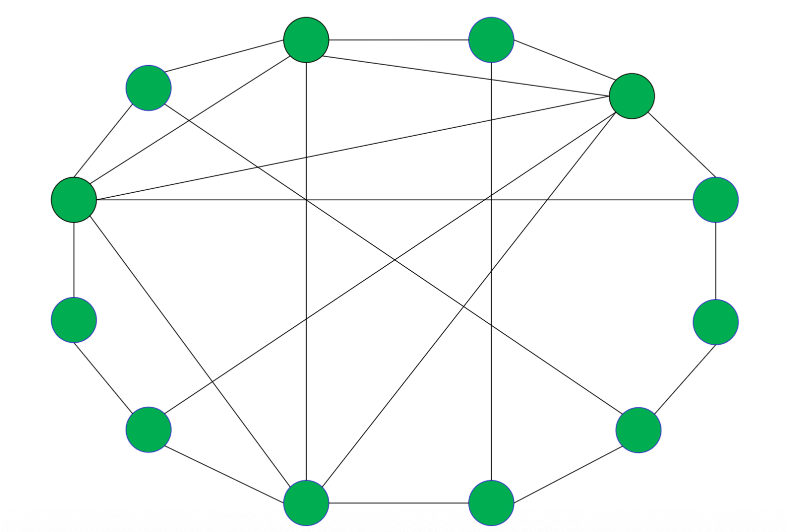


Figure 11: Eq. of this shirt game

That is, all nodes will choose solid in the end.

But why is this game a perfect example of Example 3.3.1? If we try to calculate the true social average *preference for solid shirts*: $\frac{4}{12} = \frac{1}{3}$.

Then we calculate each nodes perceived average *preference for solid shirts*:

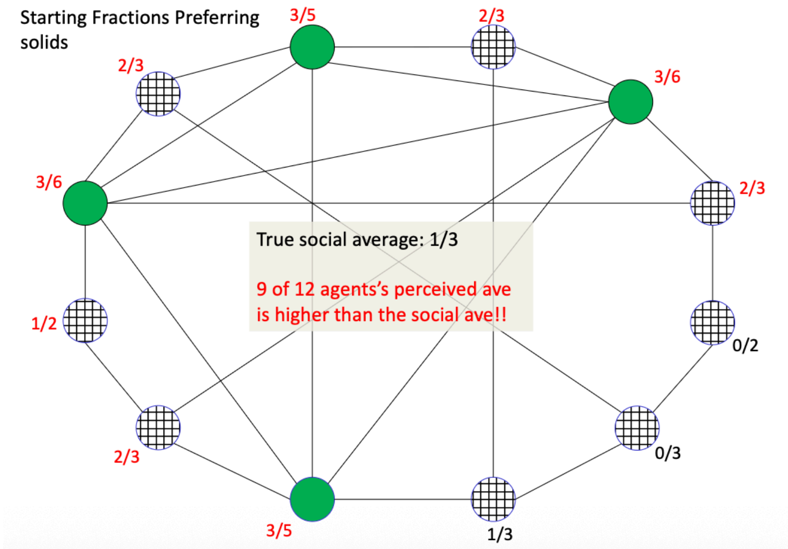


Figure 12: Perceived averages

It turns out that 9 of 12 agents's perceived ave is higher than the social ave!!

Because your friends are often more "socially active" than you (the friendship paradox), the world you see is distorted by these "socially active" people, and you mistakenly believe that their behavior represents the mainstream of society (the majority illusion).

3.3.1. Empirical Observations

1. Systematic Overestimation (Perkins et al., 2005)

TABLE 1. Students' accuracy in perceiving the drinking norm at their school (comparing actual with perceived number of alcoholic drinks consumed the last time students "partied"/socialized)

Actual school norm (median drinks)	Accuracy of perceived drinking norm: Underestimate of ≥3 drinks (%)	Underestimate of 1-2 drinks (%)	Accurate estimate (%)	Overestimate of 1-2 drinks (%)	Overestimate of ≥3 drinks (%)	Total	N of respondents	N of schools
0	NA	NA	20.6	19.5	59.9	=100%	1,891	4
1	NA	10.5	3.8	28.5	57.2	=100%	2,526	6
2	NA	7.5	8.1	30.6	53.7	=100%	8,345	14
3	3.8	6.4	13.5	37.5	38.7	=100%	18,859	35
4	3.1	12.3	12.6	37.0	34.9	=100%	20,353	38
5	4.3	15.8	20.6	24.1	35.3	=100%	11,481	20
6	6.9	23.2	15.0	23.5	31.5	=100%	8,912	12
7	5.7	23.3	9.7	23.6	37.8	=100%	352	1
Total schools	3.4	11.8	13.8	31.9	39.1	=100%	72,719	130

The vast majority of people ($31.9\% + 39.1\% \approx 71\%$) seriously overestimated their classmates' alcohol consumption.

2. Positive Correlation(degree, behavior) For middle school students (12-15 years old)...

1. The probability of smoking increases by 5% for every additional friend you have. (Valente et al., 2005)
2. For every additional friend you have, the likelihood of trying alcohol increases by 6%.(Tucker et al., 2013)

Those "high-degree agents" who are at the center of the network are often the ones who smoke and drink the most.

So why?

3.3.2. Takeaways

1. Because of the friendship paradox, when you observe your surroundings, you are more likely to see people with many friends (social butterflies). They appear in your field of vision far more frequently than ordinary people.

2. Data tells us that these easily visible social media influencers are precisely the ones who drink heavily and smoke a lot.

But why does the 2nd point hold?

This is because of strategic complementarity.
1. People in central positions (high degree/centrality) are most influenced by their friends (positive feedback loop), or they must exert more effort (action) to maintain their central position.
2. In this context, to maintain a "popular" status, or encouraged by friends, socially central individuals are more likely to demonstrate themselves or integrate into the circle to some extent by "drinking/smoking more."

Actually, the 2nd one can be endogenized!! (Jackson, 2019)

3.4. Friends Paradox as per (Jackson, 2019)

Let's set this up!
- there are n players, each with intrinsic preference θ_i
- each one chooses an action x_i
- in a random network
- Let $P_i(d)$ denote the distribution of population of i's potential neighbors' degrees.

Lemma 3.4.1 (Expected Degree of Neighbors): The distribution of i's neighbors' expected degrees is given by

$$P_i(d)\frac{d}{\mathbb{E}_i[d]} = P_i(d)\frac{d}{\sum_d dP_i(d)} \tag{62}$$

, so the expected degree of i's neighbors is:

$$\hat{\mathbb{E}}_i[d] = \sum_d dP_i(d)\frac{d}{\mathbb{E}_i[d]} = \frac{1}{\mathbb{E}_i[d]}\mathbb{E}_i\left[d^2\right] = \mathbb{E}_i[d] + \frac{\mathbb{V}_i[d]}{\mathbb{E}_i[d]} > \mathbb{E}_i[d] \tag{63}$$

Then we can use this lemma to get a proposition

Proposition 3.4.1: In a game with strategic complements, we have

$$x_i^{\text{prec}}(\theta_i, d_i) \geq x_i^{\text{soc}}(\theta_i, d_i), \forall \theta_i, d_i \tag{64}$$

and

$$\underbrace{\hat{\mathbb{E}}\left[x_i^{\text{prec}}(\theta_i, d_i)\right] \geq}_{\text{network perception effect}} \underbrace{\mathbb{E}\left[x_i^{\text{prec}}(\theta_i, d_i)\right] \geq \mathbb{E}\left[x_i^{\text{soc}}(\theta_i, d_i)\right]}_{\text{network eq. effect}} \tag{65}$$

For Equation 64, we explain it by that mistakenly believing others' behavior is superior → You also improve your own behavior.

For Equation 65,
1. network perception effect: *You believe others are doing more than they actually are.*
2. network eq. effect: *Strategic complementarity allows you to do more in eq.*

So in general, we interpret Proposition 3.4.1 as **Because highly observed nodes are over-observed and there is strategic complementarity in behavior, individuals perceive their neighbors' behavior as higher than the actual social behavior, and this illusion in turn inflates the behavior in equilibrium.**

And *Welfare up or down, depending on externalities*

3.5. Some Backups

3.5.1. More on (Bramoullé & Kranton, 2007)

1. Any maximal independent set is an eq.
2. How to select one?

- idea: Stability as Equilibrium Selection

How to interpret? *An equilibrium is stable; if you push it slightly, it will automatically return to its original equilibrium. An unstable equilibrium, on the other hand, will be swayed to a different equilibrium by even the slightest disturbance.*

Consider an eq. x, and

$$x^0 = \begin{pmatrix} x_1 + \varepsilon_1 \\ x_2 + \varepsilon_2 \\ \vdots \\ x_n + \varepsilon_n \end{pmatrix} \tag{66}$$

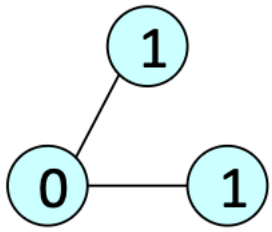
where ε_i is a small perturbation.

Then let x^1 be the best response to x^0, and iterate forward:

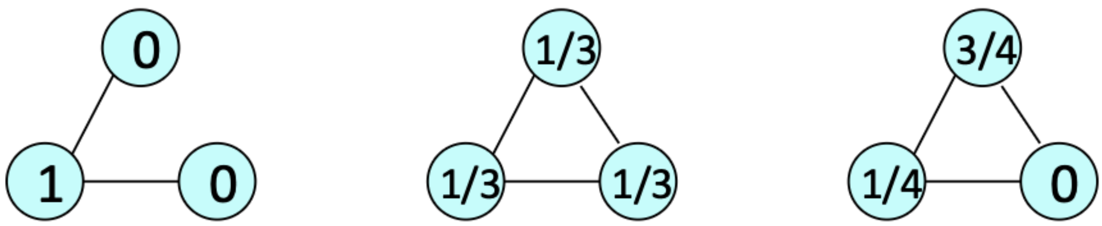
$$x^t = \mathrm{BR}\left(x^{t-1}\right) \tag{67}$$

and if after T periods the whole system goes back to x then we say it's stable.

So in the BK model, we can find that the *specialist equilibria* is stable, that is



this one is good but these:



are not. And the proposition for this is

Proposition 3.5.1: Only stable equilibria are specialist equilibria such that every non-specialist has at least two specialists in his or her neighborhood

But why?

1. If there is only one specialist, when the specialist's level drops slightly, the free-rider will start providing its own services.

2. If there are at least 2 specialists, even with minor perturbations:
   1. The free-rider still sees a neighbor that is working hard enough.
   2. It continues free-riding (remaining at 0).
   3. The specialist will also remain at 1 under the best-response condition.
   4. The system returns to equilibrium → stable.

3.5.2. (Jackson, 2019) Details

3.5.2.1. Linear Quadratic Game

Let's construct this social expectation utility:

$$\mathbb{E}[U_i^{\text{soc}}(x_i, x_{-i})] = \theta_i x_i + a x_i d_i \mathbb{E}\left[x_j\right] - \frac{c}{2} x_i^2 + \varphi \mathbb{E}\left[\sum_{j \neq i} x_j\right] \tag{68}$$

and to decompose it:
1. $\theta_i x_i$: the intrinsic preference
2. $a x_i d_i \mathbb{E}\left[x_j\right]$: the strategic complementarity gain where $\mathbb{E}\left[x_j\right]$ is the average action of i's neighbors
3. $-\frac{c}{2} x_i^2$: quadratic cost
4. $\varphi \mathbb{E}\left[\sum_{j \neq i} x_j\right]$: *Under the social benchmark (SOC), the planner is concerned with the overall level of behavior, not just the utility function of individuals.* That is, Your friends also generate positive externalities for others (not you), so social planners calculate the sum of everyone's actions when optimizing.

But the perceived one is:

$$\begin{aligned} \mathbb{E}\left[U_i^{\text{prec}}(x_i, x_{-i})\right] &= \theta_i x_i + a x_i d_i \hat{\mathbb{E}}\left[x_j\right] - \frac{c}{2} x_i^2 + \varphi \mathbb{E}\left[\sum_{j \neq i} x_j\right] \\ &= \mathbb{E}[U_i^{\text{soc}}(x_i, x_{-i})] + a x_i d_i \left(\hat{\mathbb{E}}\left[x_j\right] - \mathbb{E}\left[x_j\right]\right) \end{aligned} \tag{69}$$

and so *Individuals overestimated the behavioral level of their neighbors.*

And each agent i solves:

$$\max_{x_i} \quad \theta_i x_i + a x_i d_i \hat{\mathbb{E}}\left[x_j\right] - \frac{c}{2} x_i^2 \tag{70}$$

and best response is given simply by:

$$x_i^* = \frac{1}{c}\left(\theta_i + a d_i \hat{\mathbb{E}}\left[x_j\right]\right) \tag{71}$$

so Equation 64 holds obviously.

And we focus on this:

Lemma 3.5.1: We assume the following holds:

1. $\hat{\mathbb{E}}_i[d] = \hat{\mathbb{E}}[d]$, that is *Everyone has the same perception of the average degree of their neighbors.*
2. and $c > a\hat{\mathbb{E}}[d]$: *Only when the cost is high enough and the complementarity is not too strong can an eq. be guaranteed without causing an explosion and to remain unique.*

then there exists a unique eq.:

$$x_i^{\text{prec}}(\theta_i, d_i) = \frac{\theta_i}{c} + \frac{ad_i\hat{\mathbb{E}}[\theta]}{c^2 - ca\hat{\mathbb{E}}[d]} \tag{72}$$

so how to get the Lemma 3.5.1? we consider br as in Equation 71, then

$$\begin{aligned} x_i &= \frac{1}{c}(\theta_i + ad_i\hat{\mathbb{E}}[x]) \\ \hat{\mathbb{E}}[x] &= \hat{\mathbb{E}}\left[\frac{1}{c}(\theta_i + ad_i\hat{\mathbb{E}}[x])\right] \\ \hat{\mathbb{E}}[x] &= \frac{1}{c}[\hat{\mathbb{E}}[\theta] + a\hat{\mathbb{E}}[d]\hat{\mathbb{E}}[x]] \end{aligned} \tag{73}$$

then we can get $\hat{\mathbb{E}}[x]$ and then plug it into the Equation 71 to get Equation 72.

So it's *higher actions for higher types, higher degree*. Similarly, we have

$$x_i^{\text{soc}}(\theta_i, d_i) = \frac{\theta_i}{c} + \frac{ad_i\mathbb{E}[\theta]}{c^2 - ca\mathbb{E}[d]} \tag{74}$$

so by Lemma 3.4.1, Equation 64 holds.

Moreover, let's introduce some stats thing.

Definition 3.5.1 (First-Order Stochastic Dominance): Let the CDF of X and Y be $F_X(x) = \Pr\{X \le x\}$ and $F_{Y(y)} = \Pr\{Y \le y\}$, then we say $X \underset{\text{FOSD}}{\succ} Y$ iff

$$F_X(x) \le F_Y(x), \forall x \in \mathbb{X} \tag{75}$$

and $<$ holds strictly for some x.

Then by definition

$$\hat{P}_i(d) \equiv P_i(d)\frac{d}{\mathbb{E}[d]} \underset{\text{FOSD}}{\succ} P_i(d) \tag{76}$$

and therefore

$$\hat{\mathbb{E}}[d] > \mathbb{E}[d] \tag{77}$$

4. Cooperation on Networks

4.1. Lessons from Sociology (Coleman, 1988)

When people in a social network know each other (closed structure), it is easier for them to form common norms, build trust, and restrain an individual's bad behavior through collective pressure.

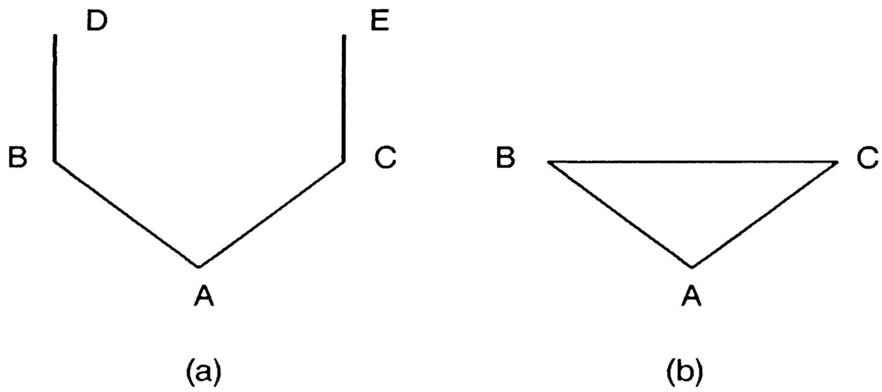


FIG. 1.—Network without (*a*) and with (*b*) closure

This is the so called *collective sanction*.

> **Definition 4.1.1** (Collective sanction) : Collective sanction refers to the collective pressure, punishment, or constraint exerted on an individual by members of a group in order to establish norms.

4.1.1. Going back to Clustering Coefficients

1. Prison friendships
   - .31 (MacRae 60) vs .0134
2. Co-authorships
   - .15 math (Grossman 02) vs .00002,
   - .09 biology (Newman 01) vs .00001,
   - .19 econ (Goyal, van der Leij, Moraga 06) vs .00002,
3. Web
   - .11 for web links (Adamic 99) vs .0002

We live in a highly clustered network. A friend's friend is likely to be a friend as well.*It is precisely because of such a high clustering coefficient that effective norms, trust, and collective sanctions can be formed in society.*

4.2. Formalizing with Incentives: Cooperation on Networks

The game is called *Favor exchange*

1. cost of giving a favor: c
2. net benefit per period is v
3. total net benefit with discount factor δ:

$$V = \sum_{t=1}^{\infty} \delta^t v = \frac{\delta}{1-\delta} v \tag{78}$$

Self-enforcing This refers to cooperation that can be maintained solely through mutual interests, without the need for police or laws.

In this case, it's:

$$c < V = \frac{\delta}{1-\delta} v \tag{79}$$

4.2.1. Why network matters here

So if it's a one to one relation ship, we need $c < V = \frac{\delta}{1-\delta} v$,but if we have a network like this:

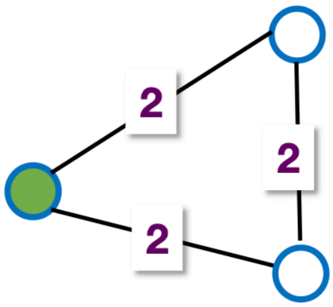


Figure 16: Community-enforcement

In Figure 16, we have one mechanism: **Ostracize deviating agent**

If the green dot does something bad (such as refusing to help one of the white dots). Because the two white dots are friends, they share information. If one of the green dots betrays the other, **both white dots will punish the green dot simultaneously** (for example, by severing their relationship with it).

So in this case, we have

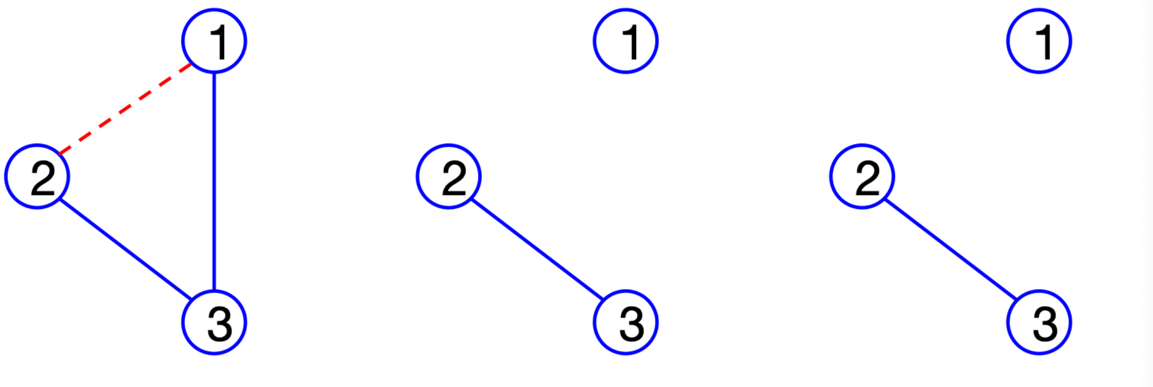
$$c < 2 \times V = 2 \times \frac{\delta}{1-\delta} v \tag{80}$$

4.3. (Jackson et al., 2012)

Ostracism without cost?

If a traitor is kicked out as punishment, the remaining group may be **too small to maintain cooperation**, ultimately causing the entire network to collapse.

Suppose $V < c < 2V$:



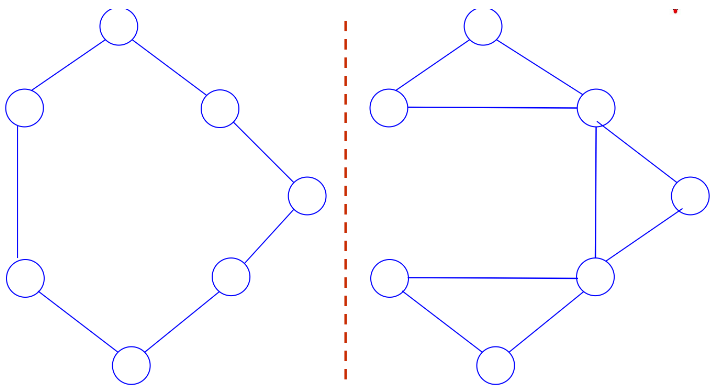
In this case, first 1 deviates against 2, then *1 is ostracized*. But when 3 deviates against 1, she will find that maintain the relationship with 2 is no longer profitable, so the entire network collapses.

Therefore, this threat of punishment is non-credible.

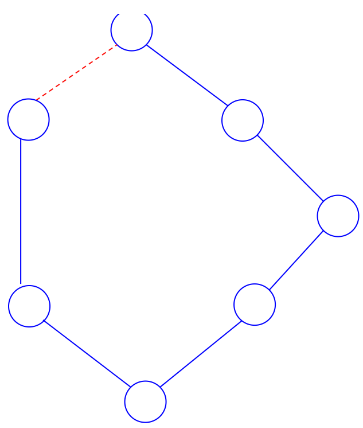
Formally, at period t:

1. At most one agent i_t is called upon to perform a favor for $j_t \in N_i(g_t)$
2. i_t keeps or deletes the link
3. others can respond: announce which (remaining) links they wish to maintain
4. links are retained iff mutually agree
5. the resulting network is g_{t+1}

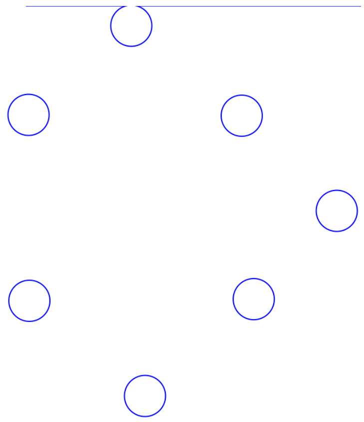
We consider 2 structures:



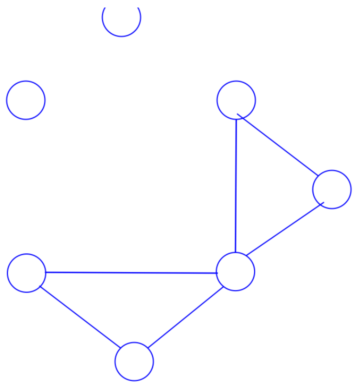
For the left structure, if the there is one disconnection:



Then it eventually leads to totally separated nodes:



that is, **autarchy**. But for the right structure, suppose the same split happens, it will end up as:



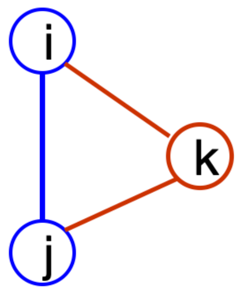
That is, it ends up losing a triad. One observation here is that the right structure preserves more than the left autarchy case.

Definition 4.3.1 (Robustness Against Contagion): A network such that the punishment for failing to perform a favor only impacts neighbors of original players lose links

Deleting a node or experiencing a disturbance (default) has a *localized* impact.

Definition 4.3.2 (Supported Link): Link $ij \in g$ is supported if there exists k such that $ik \in g, jk \in g$.

That is, i, j, k forms a triangle.



It's interesting to bring out the theorem:

Theorem 4.3.1: If no pair of players could sustain favor exchange in isolation and a network is robust, then all of its links are supported.

Supported: In online games, this usually means that the existence of an edge is not only due to the relationship between two people, but is also monitored and supported by a "third party" or "mutual friend" (e.g., if A betrays B, C will also punish A).

4.4. Closure vs. Support

1. Closure (Coleman, 1988)
2. Support (Jackson et al., 2012)

Any differences?

For closure, we need to ask a question as in Figure 22, that is w/ what frequency are a typical node A's neighbors, say B and C, neighbors of each other?

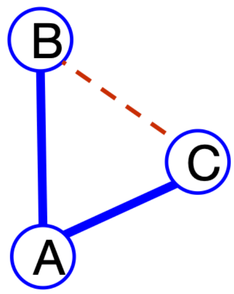


Figure 22: closure

And for support, as in Figure 23, we wonder w/ what frequency do a typical pair of connected nodes, say A and B, have a common neighbor C?

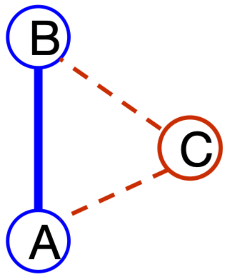
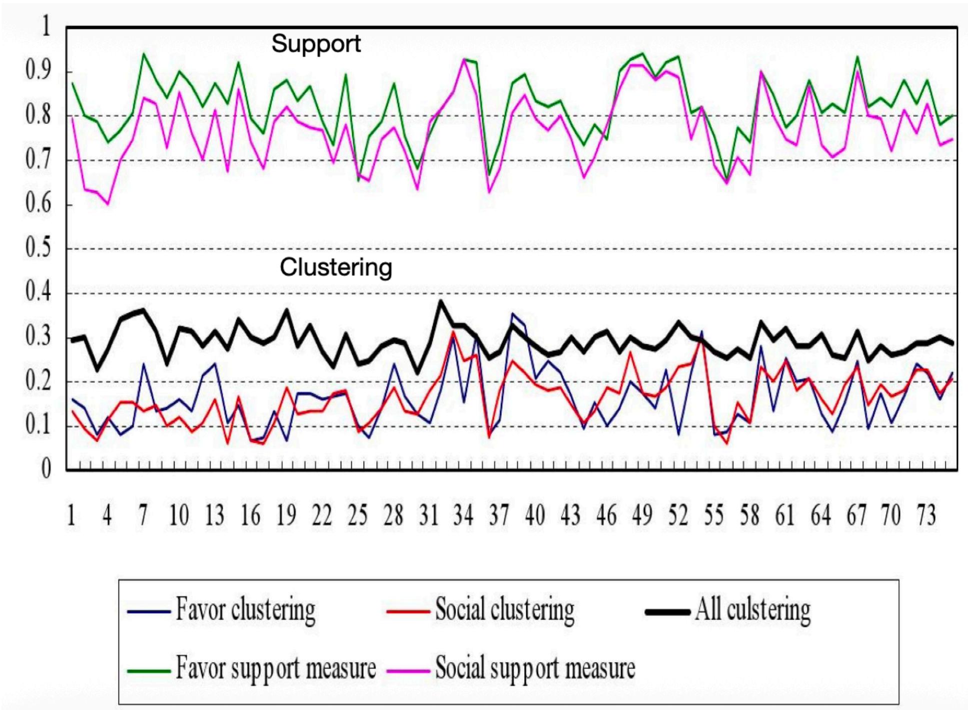


Figure 23: Support

4.5. Empirical Tests (Banerjee et al., 2013)

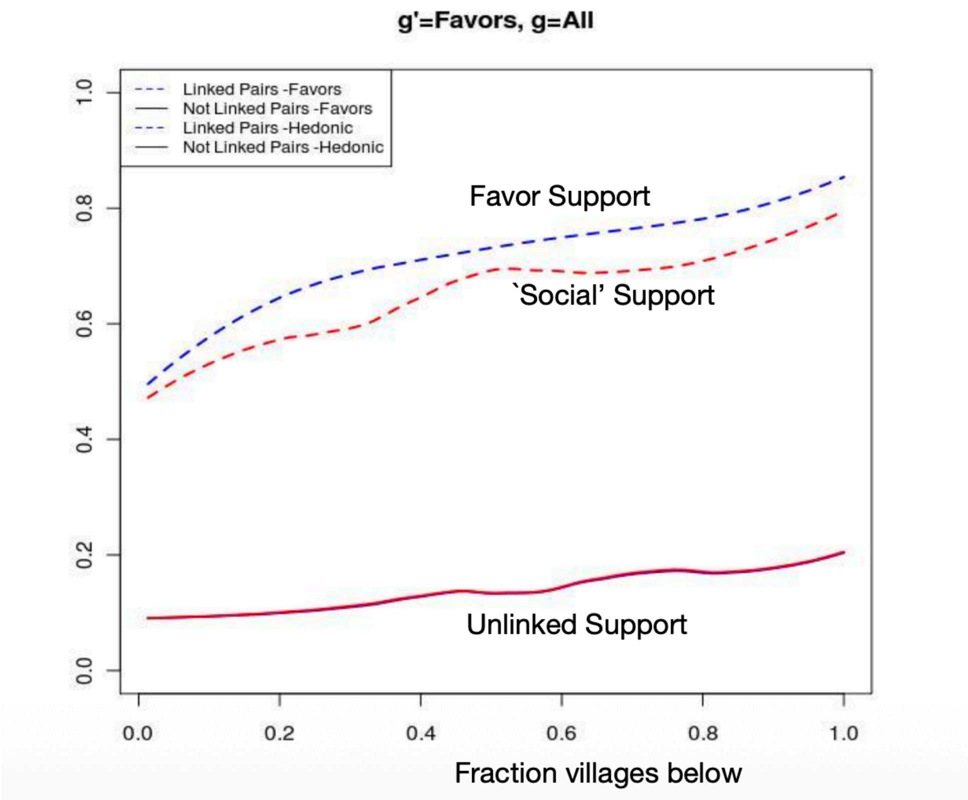
1. Researchers surveyed 75 rural villages in Karnataka, India.
2. A microfinance agency (BSS) entered 43 of these villages.
3. Before the lending institutions even entered the field, researchers had already mapped out the network of relationships among the villagers. The key point is that they distinguished between two different types of relationships:
   1. "**Favor**" **Networks**: These involve the exchange of favors, such as lending money, rice, or kerosene.[6]
   2. "**Social**" **Networks**: Purely for chatting, visiting others' spaces, and maintaining friendships. (Low risk).



- Support: *This means that if two people have a loan relationship (favor link) in these villages, there is an 80%-90% probability that they have at least one mutual friend.*
- Clustering: *This means that two friends of one person do not necessarily know each other.*

And here's another figure:

[6]Note: These relationships are risky; the money may not be repaid, so theoretically, "Support" is necessary.



Having mutual friends is a very strong predictor of establishing a lending relationship. I'm only willing to lend you money because we have mutual friends.

"Profit-driven structure"

- Because we need to lend and borrow from each other and are afraid that the other party will default (demand),
- we deliberately build a "triangle" relationship network (structure) with a large number of mutual friends.
- This structure is most accurately measured by the "Support" indicator and looks like a "patchwork" (form).

4.6. Multiplexity

Multiplex to better sustain cooperation

- prisoner's dilemma: one may deviate against a partner
  - like not return money borrowed
- incentives to prevent such deviations:
  - potential interactions in future on the same link
  - punishments from other (common) friends
- **multiplex**: deviation on one relation is punished by other relations

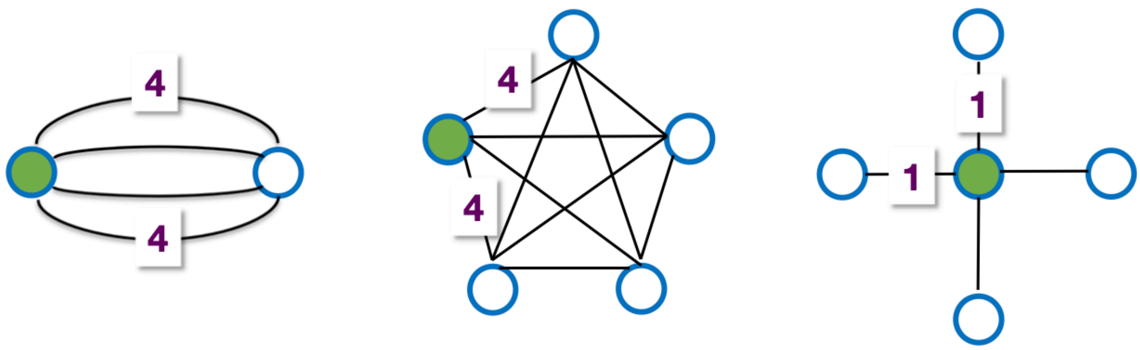


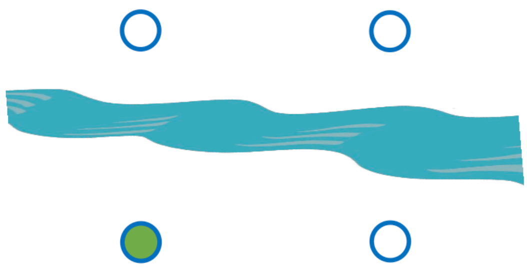
Figure 26: 3 types of networks

multiplex and community enforcement both enhance cooperation
- multiplex is more robust: rely less on the rest of the network

In Figure 26, we have
- left: multiplex
- middle: community enforcement
- right: star network w/o support

4.6.1. Multiplex trap and homophily

Think about the environment:



so there are 2 types of relations:
1. kid care: better on one side
2. risk share: better across the river

For the green node, she will link kid care w/ the lower right node
1. then she'll decide the risk-share link
2. it's better to link the upper left one
3. but w/ multiplexity, she'll feel it better to link w/ the same lower right node:

In the end, we have the eq. as in the left of Figure 28.
- homophily expands from one relation to other relations
- so called. **multiplex trap**

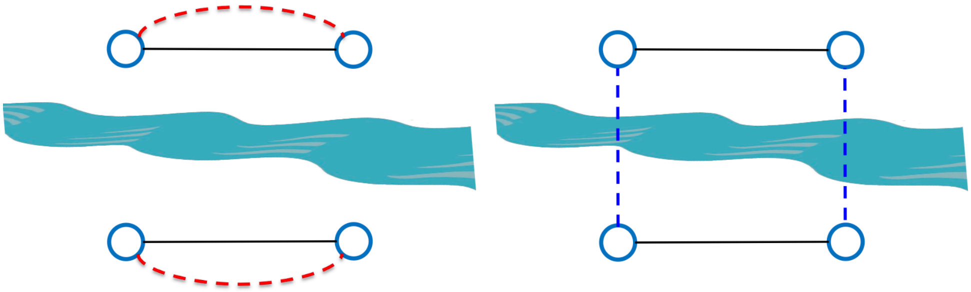


Figure 28: Eq. and efficiency

The most efficient outcome in the right side of Figure 28: not an eq.!

5. Financial networks

5.1. (Allen & Gale, 2000)

- Viewed as a starting point
- An arrow: one holds deposits in another bank

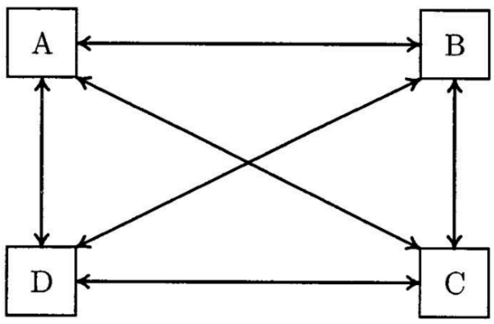


FIG. 1.—Complete market structure

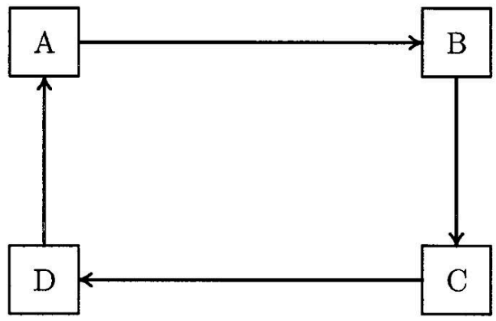


FIG. 2.—Incomplete market structure

The left one is viewed as “complete” and the right one is “incomplete”.

Consider a shock to A and makes it bankrupt:

- in the right figure:

$$D \to C \to B \tag{81}$$

- in the left figure: equally shared by $B, C, D \Longrightarrow$ More likely they all survive.

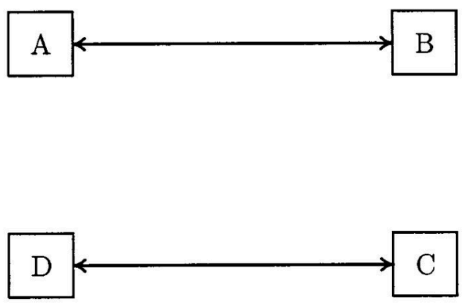


FIG. 3.—Disconnected incomplete market structure

Consider a shock that makes A bankrupt:

- only affect B
- less fragile than fig 2

5.2. (Acemoglu et al., 2015)

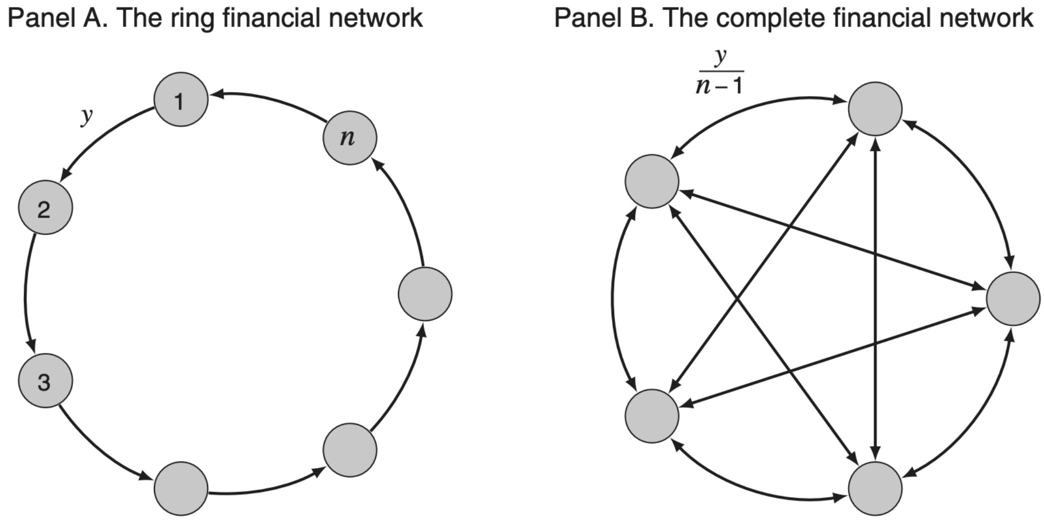


FIGURE 1. THE RING AND THE COMPLETE FINANCIAL NETWORKS

- Arrow: debt contracts

For a **small** negative shock, the ring (left one) is fragile.
- more like a ring: more fragile
- more complete: more stable

For a **large** negative shock:
- **Ring and complete networks are the least stable ones**

but once we have a separation "δ-connected":
- when δ is small, this is more stable than ring or complete.

Phase Transition: under small shocks vs. large shocks, the more stable network structures can be very different

- Pros:
  - generalization to (Allen & Gale, 2000)
  - debt contracts: widely accepted by macro/finance audience
- Cons:
  - model not very tractable
  - limited lessons regarding network structures

5.3. (Elliott et al., 2014)

- develop network-based measures of cascades
- distinguish the effects of *diversification* and *integration*.
- Highlight non-monotonic effects of diversification and integration on contagions.
- Offer a simple illustration of how the model can be used empirically.

5.3.1. Basics

- organizations have claims on
  - fundamental assets
  - other organizations
- Bankruptcy cost:
  - when an organization's value falls below a level, the value of other's claims on its drop discontinuously
  - drop in value of one leads to drop in values of others: **cascades**

5.3.2. Model

- organizations: $\{1, 2, \ldots, n\}$
- Assets: $\{1, 2, \ldots, m\}$
- let price of asset k be p_k
- and let D_{ik} be the holdings of asset k by organization i

Value of an organization:
- book value:

$$V_i = \sum_{k=1}^{m} D_{ik} p_k + \sum_{j=1}^{n} C_{ij} V_j \tag{82}$$

  - where $\sum_{k=1}^{m} D_{ik} p_k$ is direct asset holdings
  - and $\sum_{j=1}^{n} C_{ij} V_j$ is cross holdings

In vector form:

$$V = Dp + CV \tag{83}$$

that is,

$$V = (I - C)^{-1} Dp \tag{84}$$

- Market Value:
  - defined as value to private investors
  - let

$$v_i = \hat{C}_{ii} V_i \tag{85}$$

  - that is:

$$v = \underbrace{\hat{C}(I - C)^{-1}}_{A} Dp \tag{86}$$

  - and then A_{ij} denotes: *fraction of the returns owned by org j that ultimately accrue to private shareholders of i*

Example 5.3.1: Let there be $n = 2$ firms, each owns half of the other:

$$C = \begin{pmatrix} 0 & 0.5 \\ 0.5 & 0 \end{pmatrix} \tag{87}$$

then it must own half of itself:

$$\hat{C} = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix} \tag{88}$$

and then

$$\begin{aligned} A = \hat{C}(I - C)^{-1} &= \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix} \begin{pmatrix} 1 & -0.5 \\ -0.5 & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix} \begin{pmatrix} \frac{4}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{4}{3} \end{pmatrix} \\ &= \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix} \end{aligned} \tag{89}$$

and this is depicted as Figure 30

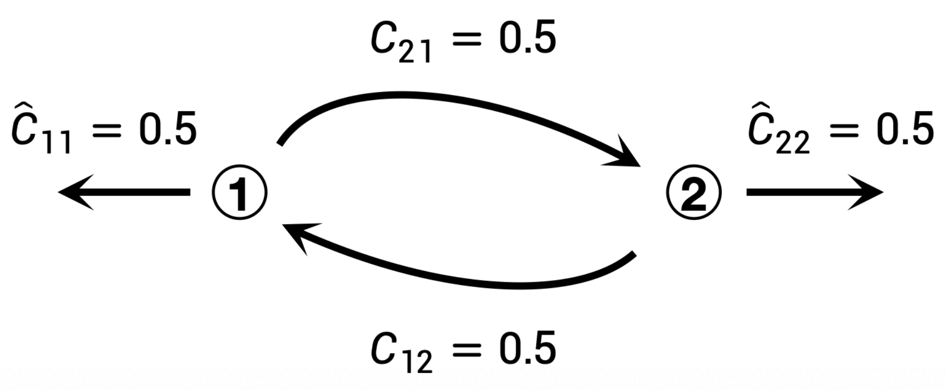


Figure 30: cascade of 2 firms

5.3.3. Bankruptcy

if an org's value drops below some *threshold* θ:

- its value falls discontinuously by β_i
- let the threshold be $\underline{v}_i$, then

$$b_i(v_i) = \begin{cases} \beta_i & \text{if } v_i < \underline{v}_i \\ 0 & \text{o.w.} \end{cases} \tag{90}$$

w/ bankruptcy, we have

$$v = A[Dp - b(v)] \tag{91}$$

Solution

- let $s_i = 1$ if an org. fails
  - $s_i = 0$ if not
- $s = (s_1, \ldots, s_n)$ is an equilibrium if

$$v = A[Dp - \text{diag}(\beta)s] \tag{92}$$

  and

$$v_i < \underline{v}_i \iff s_i = 1 \tag{93}$$

  that is

$$s(v) = \mathbb{I}(v < \underline{v}) \in \mathbb{R}^{n \times 1} \tag{94}$$

Equilibria

- Bankruptcies are complements!
  - one fails → others likely to fail
  - the set of equilibria forms a complete lattice
- exists a "base case" eq.
  - fewest orgs fail
- algorithm to find it:
  - identify orgs that fail even all others success
  - then identify failures due to the failure of the first org.

simulation

- one asset per org starting at value 1
- shock is to pick one org.'s direct investment to devalue to 0
- then check if it fails according to Section 5.3.3
- look at the result cascade

Let there be $n = 100$ org. and form a random network w/

$$\Pr\left\{g_{ij} = 1\right\} = \frac{d}{n-1} \tag{95}$$

- where d is the expected number of other org. that an org. cross holds
- and d is therefore *level of diversification*

Then we let c fraction of org cross-hold:

- so $1 - c$ holds privately

Then claim i has on j:

$$C_{ij} = c\frac{g_{ij}}{d_j} \tag{96}$$

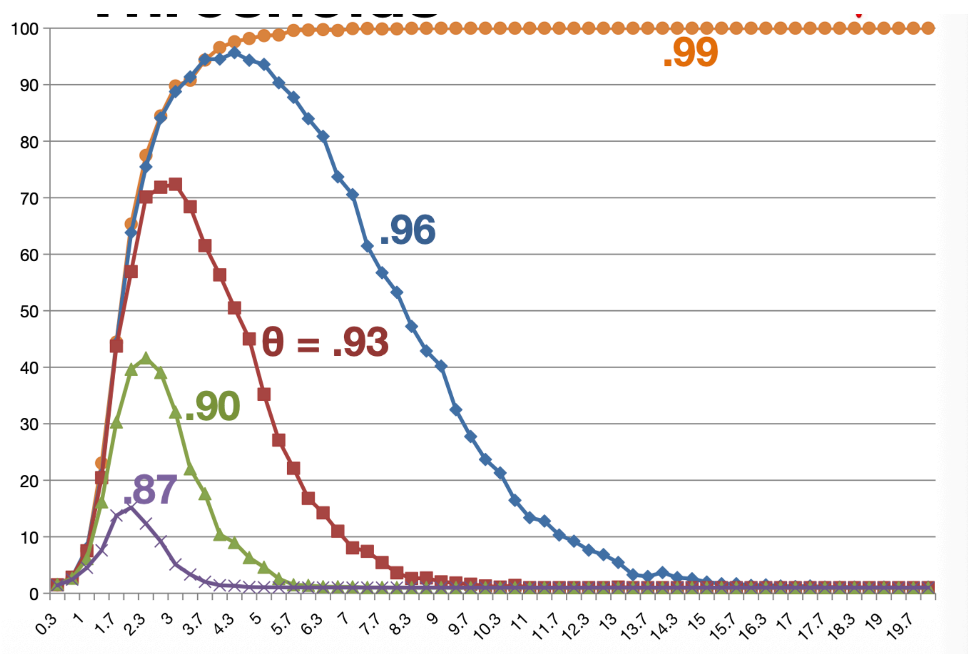


Figure 31: diversification and contagion

The x-axis of Figure 31 is the degree and the y-axis of it is the percent of orgs that fail. different lines reports different threshold θ.

A cascade

Three necessary components of it:

1. a first failure
2. contagion: that is, other org. must be exposed to the failed org. sufficiently to fail too
3. wider propagation: sufficiently large components

Diversification

- has non-monotonicity
  - increases component size
  - decrease the impact of one organization on its neighbors

So there is a *dangerous middle levels*:

- low diversification:
  - fragmented network
  - no widespread contagion
- high
  - little exposure to any single other org.
  - failures do not spread

however, in the middle:

- connected, contagion is possible
- exposure to only a few others makes it easy to spread.

Integration

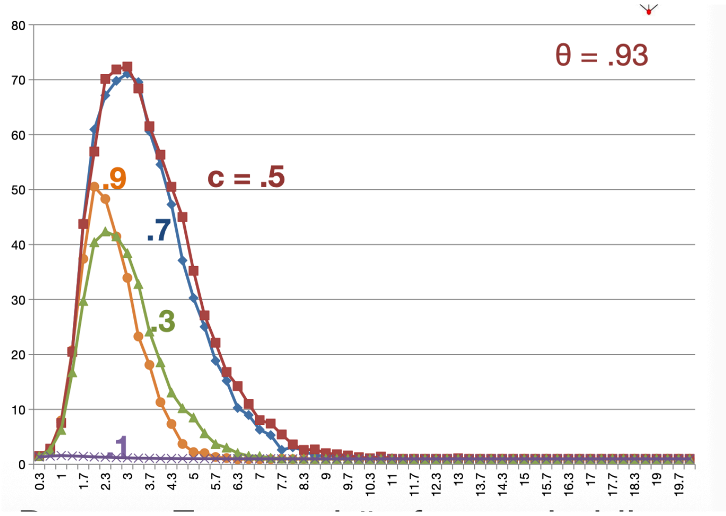


Figure 32: integration

Here the x-axis is the degree: expected number of cross holdings.

- Low:
  - little exposure to others,
  - failures don't trigger others

- high:
  - difficult to get a first failure
  - failure of own assets doesn't trigger failure

- middle:
  - exposure to others enough to trigger contagion

5.3.4. summary

- a first failure:
  - some org. needs to fail
  - *integration decreases own-asset dependence*
- initial contagion:
  - some neighbors need to be sufficiently exposed to fail too

  1. *integration increases exposure of neighbors*
  2. *diversification decreases exposure of specific neighbors*
- Interconnection:
  - to continue to cascade widely, the network must have sufficiently large components.

5.4. Eurodebt (Elliott et al., 2014)

- 6 key countries in Europe that have large cross holdings of each other's sovereign debt
- isolated system
- what happens if values fall and debt is devalued

Skip all those:

- Pros:
  - extremely tractable
  - interesting implications
  - can be taken to data
- Cons
  - cross-holding based model
  - all mechanical no economic incentives at all!

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