Toy DMP Model in CT

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This note follows a lot from the note by Toshihiko Mukoyama.

1 Unemployment Dynamics

Let the period length be Δ and unemp. rate at time t be $u(t)$:

$$u(t+\Delta) = \sigma\Delta(1-u(t)) + (1-\lambda_w\Delta)\,u(t)$$

where $\sigma\Delta > 0$ is the prob. of losing the job, $\lambda_w\Delta > 0$ be the prob. of finding a job over Δ.

We can rewrite it as:

$$\begin{aligned}
u(t+\Delta) &= \sigma\Delta - \sigma\Delta u(t) + u(t) - \lambda_w\Delta u(t) \\
u(t+\Delta) - u(t) &= \sigma\Delta - \sigma\Delta u(t) - \lambda_w\Delta u(t) \\
\frac{u(t+\Delta) - u(t)}{\Delta} &= \sigma - \sigma u(t) - \lambda_w u(t) \\
\frac{u(t+\Delta) - u(t)}{\Delta} &= \sigma(1-u(t)) - \lambda_w u(t)
\end{aligned}$$

so taking $\Delta \to 0$:

$$u'(t) = \sigma(1-u(t)) - \lambda_w u(t)$$

We then try to endogenize job-finding rate λ_w. Let agg. vacancy be v and the matching function takes the form of

$$M(u,v)\,\Delta$$

Some assumptions:

1. matching is random
   (a) all vacancies have an equal chance to match w/ an employed worker
   (b) all unemployed workers have an equal chance to match w/ a vacancy
2. M is increasing in both terms
3. M is CRS

Under these assumptions, the prob. of a vacancy matching w/ an unemp. worker during Δ is

$$\frac{M(u,v)\Delta}{v} = M\left(\frac{1}{\theta},1\right)\Delta = \lambda_f(\theta)\Delta$$

where $\theta \equiv \frac{v}{u}$ is the tightness. The prob. of an unemp. worker matching w/ a vacancy during Δ is:

$$\frac{M(u,v)\Delta}{u} = M(1,\theta)\Delta = \lambda_w(\theta)\Delta$$

and $\lambda_w(\theta) = M(1,\theta) = \theta M\left(\frac{1}{\theta},1\right) = \theta\lambda_f(\theta)$. So unemp. rate dynamics now become:

$$u'(t) = \sigma(1-u(t)) - \lambda_w(\theta(t))\,u(t)$$

1.1 Beveridge curve condition

Consider ss. whre $v(t) \equiv v, \forall t$ and $u'(t) = 0$:

$$\begin{aligned}
0 &= \sigma(1-u) - \lambda_w\left(\frac{v}{u}\right)u \\
\lambda_w\left(\frac{v}{u}\right)u &= \sigma(1-u) \\
u\left(\sigma + \lambda_w\left(\frac{v}{u}\right)\right) &= \sigma \\
u &= \frac{\sigma}{\sigma + \lambda_w\left(\frac{v}{u}\right)}
\end{aligned}$$

This $u = \frac{\sigma}{\sigma+\lambda_w\left(\frac{v}{u}\right)}$ is referred to as the Beveridge curve condition.

2 DMP Model

2.1 Workers and Firms

The DMP builds on the unemp. dynamics and adds the endo. determination of agg. vacancy $v(t)$. Firms create vacancies by trading off the current vacancy posting cost and future profit from the firm-worker match.

We assume that each vacancy hires one worker. Let $w(t)$ be the eq. wage. The PV of being employed, in discrete-time setting w/ period length Δ is

$$W(t) = w(t)\Delta + \frac{1}{1+r\Delta}\left((1-\sigma\Delta)W(t+\Delta) + \sigma\Delta U(t+\Delta)\right) \tag{1}$$

where $U(t)$ is the value of being unemployed.

$$U(t) = b\Delta + \frac{1}{1+r\Delta}\left(\lambda_w(\theta(t))\Delta W(t+\Delta) + (1-\lambda_w(\theta(t))\Delta)U(t+\Delta)\right)$$

Multiplying $1+r\Delta$ on both sides of (1) we get:

$$\begin{aligned}
(1+r\Delta)W(t) &= (1+r\Delta)w(t)\Delta + \left((1-\sigma\Delta)W(t+\Delta) + \sigma\Delta U(t+\Delta)\right)\\
r\Delta W(t) &= (1+r\Delta)w(t)\Delta + \left((1-\sigma\Delta)W(t+\Delta) + \sigma\Delta U(t+\Delta)\right) - W(t)\\
rW(t) &= (1+r\Delta)w(t) + \sigma U(t+\Delta) + \left(\frac{1}{\Delta} - \sigma\right)W(t+\Delta) - \frac{1}{\Delta}W(t)\\
rW(t) &= (1+r\Delta)w(t) + \sigma\left(U(t+\Delta) - W(t+\Delta)\right) + \frac{W(t+\Delta) - W(t)}{\Delta}
\end{aligned}$$

so taking $\Delta \to 0$:

$$rW(t) = w(t) + \sigma\left(U(t) - W(t)\right) + W'(t)$$

Similarly for $U(t)$:

$$\begin{aligned}
(1+r\Delta)U(t) &= (1+r\Delta)b\Delta + \left(\lambda_w(\theta(t))\Delta W(t+\Delta) + (1-\lambda_w(\theta(t))\Delta)U(t+\Delta)\right)\\
r\Delta U(t) &= (1+r\Delta)b\Delta + \left(\lambda_w(\theta(t))\Delta W(t+\Delta) + (1-\lambda_w(\theta(t))\Delta)U(t+\Delta)\right) - U(t)\\
rU(t) &= (1+r\Delta)b + \left(\lambda_w(\theta(t))W(t+\Delta) - \lambda_w(\theta(t))U(t+\Delta)\right) + \frac{U(t+\Delta) - U(t)}{\Delta}
\end{aligned}$$

so taking $\Delta \to 0$:

$$rU(t) = b + \lambda_w(\theta(t))(W(t) - U(t)) + U'(t)$$

Because of the one-firm to one-worker assumption, the firm is either matched or vacant, the value of a matched firm is:

$$J(t) = p(t)\Delta - w(t)\Delta + \frac{1}{1+r\Delta}(\sigma\Delta V(t+\Delta) + (1-\sigma\Delta)J(t+\Delta))$$

where $p(t)\Delta$is the product produced by the match and $V(t)$ is the value of vacancy, computed by:

$$V(t) = -\kappa\Delta + \frac{1}{1+r\Delta}\left(\lambda_f(\theta(t))\Delta J(t+\Delta) + \left(1-\lambda_f(\theta(t))\Delta\right)V(t+\Delta)\right)$$

where $\kappa\Delta$ is the cost of posting a vacancy. It's then easy to see that

$$rJ(t) = p(t) - w(t) + \sigma(V(t) - J(t)) + J'(t)$$

and

$$rV(t) = -\kappa + \lambda_f(\theta(t))(J(t) - V(t)) + V'(t) \tag{2}$$

We assume that anyone can start a firm by posting a vacancy

1. if $V(t) > 0$, new entrants would keep posting vacancies until $V(t)$ becomes 0. Additional vacancies would drop the value of $V(t)$because $\lambda_f(\theta(t))$ is a decreasing function of $\theta(t) = \frac{v(t)}{u(t)}$. Therefore $V(t) = 0$ has to always hold, as long as $v(t) > 0$. [1]

2. We call the condition

$$V(t) = 0$$

the *free-entry condition*

One can formalize 1 as follows:

Let $u(t)$ be the state var. at time t, which evolves as

$$u' = \sigma(1-u) - \lambda_w u$$

[1] It's possible that $V(t) < 0$ when $v(t) = 0$.

and $v(t)$be the control var. and $v \geq 0$.Given (2) and taking $u(t), J(t), V'(t)$as given, we have

$$V(t) = \frac{\lambda_f(\theta(t)) J(t) - \kappa + V'(t)}{r + \lambda_f(\theta(t))}$$

so

$$\frac{\partial V}{\partial \lambda_f} = \frac{J(t)\left(r + \lambda_f(\theta(t))\right) - \left(\lambda_f(\theta(t)) J(t) - \kappa + V'(t)\right)}{\left(r + \lambda_f(\theta(t))\right)^2}$$

at ss., $V'(t) = 0$, so we have

$$\begin{aligned}\frac{\partial V}{\partial \lambda_f} &= \frac{rJ(t) + \kappa - V'(t)}{\left(r + \lambda_f(\theta(t))\right)^2} \\ &= \frac{rJ(t) + \kappa}{\left(r + \lambda_f(\theta(t))\right)^2} > 0\end{aligned}$$

Given the matching function, it's easy to see that $\lambda_f(\theta) = M\left(\frac{1}{\theta}, 1\right)$ is decreasing in θ:

$$\frac{\partial \lambda_f}{\partial v} = \lambda_f'(\theta)\frac{1}{u} < 0$$

so

$$\frac{\partial V}{\partial v} = \frac{\partial V}{\partial \lambda_f}\frac{\partial \lambda_f}{\partial v} < 0$$

Therefore, if $V(t) > 0$, then it follows that vacancy goes up and in turn pushing $V(t) \to 0$. If $V(t) < 0$, then nobody is willing to post a vacancy and in turn $v(t) = 0$. Finally if $V(t) = 0$, then entrants are indifferent and therefore $\forall v(t) \geq 0, s.t. V(v(t)) = 0$ will work.

2.2 Wage Determination

In this framework, for a worker to meet w/ another firm, the worker has to endure the cost of unemployment. Similarly, for the firm, losing a worker would imply waiting for another worker by incurring the vacancy cost. Thus, the matched worker and firm are in a bilateral monopoly situation.

Here we assume that the surplus is split by the level of wage that solves the following maximization problem:

$$\max_w \left(\tilde{W}(w,t) - U(t)\right)^\gamma \left(\tilde{J}(w,t) - V(t)\right)^{1-\gamma} \tag{3}$$

where $\gamma \in (0,1)$. The functions $\tilde{W}(w,t)$ and $\tilde{J}(w,t)$are the values of employed workers and a matched firm, here we make it explicit that only w in this period can move. The solution to this problem is often called *Generalized Nash Bargaining Rule*.

The functions $\tilde{W}(w,t)$ and $\tilde{J}(w,t)$ can be written as:

$$\tilde{W}(w,t) = w\Delta + \frac{1}{1+r\Delta}\left((1-\sigma\Delta)W(t+\Delta) + \sigma\Delta U(t+\Delta)\right)$$

and

$$\tilde{J}(w,t) = p(t)\Delta - w\Delta + \frac{1}{1+r\Delta}\left(\sigma\Delta V(t+\Delta) + (1-\sigma\Delta)J(t+\Delta)\right).$$

By taking FOC for (3), we have:

$$\begin{aligned}
&\max_w \gamma \ln\left(\tilde{W}(w,t) - U(t)\right) + \left(1-\gamma\right)\ln\left(\tilde{J}(w,t) - V(t)\right)\\
&0 = \frac{\gamma}{\tilde{W}(w,t) - U(t)}\frac{\partial \tilde{W}(w,t)}{\partial w} + \frac{1-\gamma}{\tilde{J}(w,t) - V(t)}\frac{\partial \tilde{J}(w,t)}{\partial w}
\end{aligned}$$

Given $\frac{\partial \tilde{W}(w,t)}{\partial w} = \Delta$, $\frac{\partial \tilde{J}(w,t)}{\partial w} = -\Delta$, we have

$$\begin{aligned}
0 &= \frac{\gamma}{\tilde{W}(w,t) - U(t)}\Delta - \frac{1-\gamma}{\tilde{J}(w,t) - V(t)}\Delta\\
0 &= \frac{\gamma}{\tilde{W}(w,t) - U(t)} - \frac{1-\gamma}{\tilde{J}(w,t) - V(t)}\\
\frac{1-\gamma}{\tilde{J}(w,t) - V(t)} &= \frac{\gamma}{\tilde{W}(w,t) - U(t)}\\
\left(1-\gamma\right)\left(\tilde{W}(w,t) - U(t)\right) &= \gamma\left(\tilde{J}(w,t) - V(t)\right) \qquad (4)
\end{aligned}$$

so the w^*that satisfies (4) is the eq. wage.

2.3 Steady State Equilibrium

First, consider the ss. eq. where $u(t)$ and $v(t)$ are constant. With the conditions above we have

$$\begin{aligned} rW &= w - \sigma(W-U) \\ rU &= b + \lambda_w(\theta)(W-U) \\ rJ &= p - w - \sigma(J-V) \\ rV &= -\kappa + \lambda_f(\theta)(J-V) \\ V &= 0 \\ \left(1-\gamma\right)(W-U) &= \gamma(J-V) \end{aligned}$$

in total 6 equations w/ 6 unknowns W, U, J, V, θ, w. We can then try to reduce this system to 1 to 1. It can be shown that we have

$$(1-\gamma)(p-b) - \frac{r+\sigma+\gamma\lambda_w(\bar{\theta})}{\lambda_f(\bar{\theta})}\kappa = 0.$$

This condition is then used to solve for θ and is often referred to as the *job creation condition.*

For simple math, we have

$$\begin{aligned} rV &= -\kappa + \lambda_f(\theta)(J-V) \\ 0 &= -\kappa + \lambda_f(\theta)(J-0) \\ J &= \frac{\kappa}{\lambda_f(\theta)} \end{aligned}$$

Given

$$\begin{aligned} rJ &= p - w - \sigma(J-V) \\ rJ &= p - w - \sigma J \\ J &= \frac{p-w}{r+\sigma} \end{aligned}$$

we have the free-entry condition on firm side:

$$\frac{\kappa}{\lambda_f(\theta)} = \frac{p-w}{r+\sigma}$$

Given

$$\begin{aligned}
rW &= w - \sigma(W-U) \\
rU &= b + \lambda_w(\theta)(W-U) \\
r(W-U) &= w - b - (\sigma + \lambda_w(\theta))(W-U) \\
(r+\sigma+\lambda_w(\theta))(W-U) &= w - b \\
W - U &= \frac{w-b}{r+\sigma+\lambda_w(\theta)}
\end{aligned}$$

so given $\left(1-\gamma\right)(W-U) = \gamma(J-V)$, we have

$$\begin{aligned}
\left(1-\gamma\right)\frac{w-b}{r+\sigma+\lambda_w(\theta)} &= \gamma\frac{\kappa}{\lambda_f(\theta)} \\
w - b &= \frac{\gamma}{1-\gamma}\frac{\kappa}{\lambda_f(\theta)}(r+\sigma+\lambda_w(\theta)) \\
p - w &= p - \left[\frac{\gamma}{1-\gamma}\frac{\kappa}{\lambda_f(\theta)}(r+\sigma+\lambda_w(\theta)) + b\right] \\
(r+\sigma)\frac{\kappa}{\lambda_f(\theta)} &= p - \left[\frac{\gamma}{1-\gamma}\frac{\kappa}{\lambda_f(\theta)}(r+\sigma+\lambda_w(\theta)) + b\right] \\
(r+\sigma)\kappa &= \lambda_f(\theta)\left(p-b\right) - \frac{\gamma\kappa}{1-\gamma}(r+\sigma+\lambda_w(\theta)) \\
(r+\sigma)\kappa + \frac{\gamma\kappa}{1-\gamma}(r+\sigma+\lambda_w(\theta)) &= \lambda_f(\theta)\left(p-b\right) \\
\left(1-\gamma\right)(r+\sigma)\kappa + \gamma\kappa(r+\sigma+\lambda_w(\theta)) &= \left(1-\gamma\right)\left(p-b\right)\lambda_f(\theta) \\
(r+\sigma)\kappa + \gamma\kappa\lambda_w(\theta) &= \left(1-\gamma\right)\left(p-b\right)\lambda_f(\theta) \\
\frac{r+\sigma+\gamma\lambda_w(\theta)}{\lambda_f(\theta)}\kappa &= \left(1-\gamma\right)\left(p-b\right)
\end{aligned}$$

which is exactly the job creation condition. To explain this:

$$\underbrace{\frac{r+\sigma+\gamma\lambda_w(\theta)}{\lambda_f(\theta)}}_{\text{annuitized job creation cost}}\kappa = \underbrace{(1-\gamma)(p-b)}_{\text{firm's share of match surplus}}$$

2.4 Transition Dynamics

Now let's consider the transition dynamics by not imposing ss. conditions. Collect:

$$\begin{aligned}
rW(t) &= w(t)+\sigma(U(t)-W(t))+W'(t)\\
rU(t) &= b+\lambda_w(\theta(t))(W(t)-U(t))+U'(t)\\
rJ(t) &= p(t)-w(t)+\sigma(V(t)-J(t))+J'(t)\\
rV(t) &= -\kappa+\lambda_f(\theta(t))(J(t)-V(t))+V'(t)\\
(1-\gamma)\left(\tilde{W}(w,t)-U(t)\right) &= \gamma\left(\tilde{J}(w,t)-V(t)\right)\\
V(t) &= 0
\end{aligned}$$

so

$$\begin{aligned}
0 &= -\kappa+\lambda_f(\theta(t))(J(t)-0)\\
J(t) &= \frac{\kappa}{\lambda_f(\theta(t))}
\end{aligned}$$

and

$$\begin{aligned}
J'(t) &= rJ(t)-\left[p(t)-w(t)+\sigma(V(t)-J(t))\right]\\
&= (r+\sigma)J(t)-\left[p(t)-w(t)\right]\\
&= (r+\sigma)\frac{\kappa}{\lambda_f(\theta(t))}-p(t)+w(t)
\end{aligned}$$

also given $(1-\gamma)(W(t)-U(t))=\gamma(J(t)-V(t))$ holds, we have that

$$\begin{aligned}
(1-\gamma)\left(W'(t)-U'(t)\right) &= \gamma\left(J'(t)-V'(t)\right)\\
(1-\gamma)\left(W'(t)-U'(t)\right) &= \gamma J'(t)
\end{aligned}$$

and

$$\begin{aligned} W'(t)-U'(t) &= rW(t)-[w(t)+\sigma(U(t)-W(t))]-rU(t)+b+\lambda_w(\theta(t))(W(t)-U(t)) \\ &= (r+\lambda_w(\theta(t))+\sigma)(W(t)-U(t))-w(t)+b \end{aligned}$$

so

$$\begin{aligned} \frac{\gamma J'(t)}{1-\gamma} &= (r+\lambda_w(\theta(t))+\sigma)\frac{\gamma}{1-\gamma}\frac{\kappa}{\lambda_f(\theta(t))}-w(t)+b \\ \frac{\gamma J'(t)}{1-\gamma}+J'(t) &= \left[(r+\lambda_w(\theta(t))+\sigma)\frac{\gamma}{1-\gamma}\frac{\kappa}{\lambda_f(\theta(t))}-w(t)+b\right]+\left[(r+\sigma)\frac{\kappa}{\lambda_f(\theta(t))}-p(t)+w(t)\right] \\ \frac{J'(t)}{1-\gamma} &= \frac{r+\sigma+\gamma\lambda_w(\theta(t))}{1-\gamma}\frac{\kappa}{\lambda_f(\theta(t))}+b-p(t) \\ J'(t) &= \frac{r+\sigma+\gamma\lambda_w(\theta(t))}{\lambda_f(\theta(t))}\kappa-\left(1-\gamma\right)\left[p(t)-b\right] \end{aligned}$$

I've been checking this and it seems that Toshihiko's note is having the wrong sign here. Anyway let's move on. Given $J(t)=\frac{\kappa}{\lambda_f(\theta(t))}$ we have

$$\begin{aligned} J(t) &= \frac{\kappa}{\lambda_f(\theta(t))} \\ J'(t) &= \frac{-\lambda'_f(\theta(t))\theta'(t)}{\left[\lambda_f(\theta(t))\right]^2} \\ &= \frac{\kappa}{\lambda_f(\theta(t))}\underbrace{\left(-\frac{\lambda'_f(\theta(t))\theta(t)}{\lambda_f(\theta(t))}\right)}_{\equiv\eta(\theta(t))}\frac{\theta'(t)}{\theta(t)} \\ &= J(t)\eta(\theta(t))\frac{\theta'(t)}{\theta(t)} \end{aligned}$$

where $\eta(\theta(t))=-\frac{\lambda'_f(\theta(t))\theta(t)}{\lambda_f(\theta(t))}>0$ is the elasticity of matching function. So if $\theta(t)>\theta$ where θ is the ss. value of the tightness, we have $J'(t)>0$ so that $\theta'(t)>0$, similarly if $\theta(t)<\theta$, we have $\theta'(t)<0$. So that the only case where $\theta(t)$ does converge is that $\theta(t)\equiv\theta$, then it's easy to see that $J'(t)=0$ and $J(t)$ is fixed too.

Therefore, we can say that job-creation condition holds even not in ss. So given

$$\theta = \frac{v(t)}{u(t)}$$
$$\frac{r+\sigma+\gamma\lambda_w(\theta)}{\lambda_f(\theta)}\kappa = \left(1-\gamma\right)\left(p-b\right)$$
$$u'(t) = \sigma(1-u(t)) - \lambda_w(\theta(t))\,u(t)$$

one can pin down the dynamics of $(v(t), u(t), \theta)$. Now let's consider the case where $u(t) > u$, where u is given by the Beveridge curve condition:

$$u = \frac{\sigma}{\sigma + \lambda_w\left(\frac{v}{u}\right)}$$

then we have $u'(t) < 0$, similarly we have $u'(t) > 0$ when $u(t) < u$, so it's easy to establish that $u(t) \to u$ monotonically. Given $v = \theta u$, the dynamic of $v'(t)$ is similar.

3 Efficiency and Hosios condition

Notation. I use the notation of the model in my paper: worker bargaining power α, matching elasticity η, exogenous separation s, vacancy cost κ_v, and the Cobb–Douglas matching function $M(u, v) = \Psi u^{\eta} v^{1-\eta}$. Hence the result reads $\alpha = \eta$; in the baseline-notes notation of Notes 1–3 the same statement is $\gamma = \eta$, with γ the worker's Nash share. Write $p(\theta) = \theta q(\theta)$ for the job-finding rate, $q(\theta)$ for the vacancy-filling rate, and

$$\eta(\theta) \equiv -\frac{q'(\theta)\theta}{q(\theta)} \in (0,1)$$

for the matching elasticity. The derivation is in continuous time, as in Notes 1–3. For a homogeneous benchmark I write y for the flow output of a match and b for the flow value of unemployment.

3.1 Object

From the steady-state *job-creation condition* under Nash bargaining and free entry was derived. Translating to model notation ($\gamma \to \alpha$, $\sigma \to s$, $\lambda_w \to p$, $\lambda_f \to q$, $\kappa \to \kappa_v$, productivity $\to y$, benefit $\to b$):

$$\frac{\kappa_v}{q(\theta)} = \frac{(1-\alpha)(y-b)}{r+s+\alpha p(\theta)}. \tag{5}$$

This pins down the decentralized tightness θ^{dec}. The goal of this note is to derive the tightness a constrained planner would choose, θ^*, and compare the two.

3.2 Why efficiency can fail

The model so far is purely positive. Before asking *how far* θ^{dec} sits from θ^*, note *why* they differ at all. When one firm posts an extra vacancy it raises $\theta = v/u$, producing two uncompensated spillovers of opposite sign:

1. **Thick-market (trading) externality**: positive, on the *other* side. Higher θ raises $p(\theta) = \theta q(\theta)$: unemployed workers match faster. The firm is not paid for this.

2. **Congestion externality**: negative, on its *own* side. Higher θ lowers $q(\theta)$: every other firm fills more slowly. The firm bears none of this.

A third wedge comes from *pricing*: the firm sinks κ_v before matching, then Nash bargaining hands the worker share α of the surplus, so the firm recovers only $(1-\alpha)$ of the value it created (**hold-up**).

3.3 The constrained planner's problem

The planner faces the **same** matching function (it cannot hand-match) and chooses the path of vacancies to maximize discounted net output.

Employed workers produce y, the unemployed produce b at home, and each vacancy costs κ_v.

$$\max_{\{\theta_t\}_{t=0}^{\infty}} \int_0^\infty e^{-rt}\Big[\, y(1-u_t) + b\,u_t - \kappa_v \theta_t u_t \Big]\, dt$$

subject to:

$$u_t' = \underbrace{s(1-u_t)}_{\text{inflow}} - \underbrace{\theta_t q(\theta_t) u_t}_{\text{outflow}}.$$

State $x = u$, control $c = \theta$.

Form the current-value Hamiltonian with costate ξ on u:

$$\mathscr{H} = y(1-u) + bu - \kappa_v \theta u + \xi \left[s(1-u) - \theta q(\theta) u \right].$$

Since more unemployment is bad, $\xi < 0$; define the social value of an additional *employed* worker $\lambda \equiv -\xi > 0$.

By Pontryagin's maximum principle, we have the following:

$$\begin{aligned}
\frac{\partial \mathscr{H}}{\partial \theta} &= -\kappa_v u - \xi u \frac{\partial}{\partial \theta} \left[\theta q(\theta) \right] = 0 \\
\xi' &= r\xi - \frac{\partial \mathscr{H}}{\partial u} \\
u' &= s(1-u) - \theta q(\theta) u
\end{aligned}$$

Then by doing little math:

$$\begin{aligned}
\frac{\partial}{\partial \theta} \left[\theta q(\theta) \right] &= q(\theta) + \theta q'(\theta) = q(\theta) \left[1 + \frac{\theta q'(\theta)}{q(\theta)} \right] = q(\theta) \left[1 - \eta(\theta) \right] \\
\frac{\partial \mathscr{H}}{\partial u} &= -y + b - \kappa_v \theta - \xi s - \xi \theta q(\theta)
\end{aligned}$$

so

$$\begin{aligned}
-\kappa_v u &= \xi u \frac{\partial}{\partial \theta} \left[\theta q(\theta) \right] \\
&= \xi u q(\theta) \left[1 - \eta(\theta) \right] \\
-\kappa_v &= \xi q(\theta) \left[1 - \eta(\theta) \right] \\
\frac{\kappa_v}{q(\theta)} &= \lambda \left(1 - \eta(\theta) \right)
\end{aligned}$$

and

$$\begin{aligned}
\xi' &= r\xi - \left[-y + b - \kappa_v \theta - \xi s - \xi \theta q(\theta) \right] \\
&= \xi \left(r + s + \theta q(\theta) \right) + y - b + \kappa_v \theta
\end{aligned}$$

at ss. we have $\xi' = 0$ so

$$\begin{aligned} 0 &= \xi\left(r+s+\theta q(\theta)\right) + y - b + \kappa_v\theta \\ -\xi\left(r+s+\theta q(\theta)\right) - \kappa_v\theta &= y - b \\ \lambda\left(r+s+\theta q(\theta)\right) - \lambda\left(1-\eta(\theta)\right)\theta q(\theta) &= y - b \\ \lambda\left[\left(r+s+\theta q(\theta)\right) - \left(1-\eta(\theta)\right)\theta q(\theta)\right] &= y - b \\ \lambda\left[r+s+\eta(\theta)\theta q(\theta)\right] &= y - b \end{aligned}$$

so

$$\begin{aligned} \frac{\kappa_v}{q(\theta)} &= \lambda\left(1-\eta(\theta)\right) \\ &= \left[\frac{y-b}{r+s+\eta(\theta)\theta q(\theta)}\right]\left(1-\eta(\theta)\right) \\ &= \frac{\left(y-b\right)\left(1-\eta(\theta)\right)}{r+s+\eta(\theta)\theta q(\theta)} \end{aligned}$$

3.4 Comparison: the Hosios condition

Place the two job-creation conditions side by side:

$$\begin{aligned} \frac{\kappa_v}{q(\theta)} &= \frac{(1-\alpha)(y-b)}{r+s+\alpha p(\theta)}. \\ \frac{\kappa_v}{q(\theta)} &= \frac{\left(y-b\right)\left(1-\eta(\theta)\right)}{r+s+\eta(\theta)\theta q(\theta)} \end{aligned}$$

They are **identical except that the decentralized α is replaced by the elasticity** η.The two coincide for all parameter values if and only if

$$\boxed{\alpha = \eta}$$

and this is the ***Hosios condition.***

Interpretation. Each side of the market should capture a surplus share equal to its marginal contribution to producing matches. The worker's search input enters $M(u, v)$ with elasticity η; paying the worker $\alpha = \eta$ makes her internalize exactly the externality she imposes, and the congestion and thick-market effects cancel.

3.5 Signing the distortion

Then the direction of the inefficiency when $\alpha \neq \eta$ must be *proved.* Define, for a standing for either α (market) or η (planner),

$$\Phi(\theta, a) \equiv \frac{\kappa_v}{q(\theta)} - \frac{(1-a)(y-b)}{r+s+ap(\theta)}$$

where we set $\Phi(\theta, a) = 0$. Because the matching function is Cobb–Douglas, $M = \Psi u^{\eta} v^{1-\eta}$, the elasticity $\eta(\theta) \equiv -\theta q'/q = \eta$ is constant. Hence η is a θ-independent parameter, the planner's condition is $\Phi(\theta, \eta) = 0$ with η fixed, and the comparison "$\alpha = \eta$" together with the IFT below hold globally. Then

$$\frac{\partial \Phi}{\partial \theta} = -\frac{\kappa_v q'}{q^2} + \frac{(1-a)(y-b)ap'}{\left(r+s+ap\right)^2}$$

given $q' < 0, p' > 0, y > b$, we have

$$\frac{\partial \Phi}{\partial \theta} > 0$$

and also

$$\begin{aligned}
\frac{\partial \Phi}{\partial a} &= \frac{\left(y-b\right)\left(r+s+ap\right) + (1-a)(y-b)p}{\left(r+s+ap\right)^2} \\
&= \frac{\left(y-b\right)\left(r+s+ap+p-ap\right)}{\left(r+s+ap\right)^2} \\
&= \frac{\left(y-b\right)\left(r+s+p\right)}{\left(r+s+ap\right)^2} > 0
\end{aligned}$$

so that by implicit function theorem,

$$\frac{d\theta}{da} = -\frac{\partial \Phi}{\partial a} / \frac{\partial \Phi}{\partial \theta} < 0$$

so the solution θ is strictly decreasing in a, since the market uses $a = \alpha$ and the planner uses $a = \eta$, then $\alpha > \eta$ means $\theta^{dec} < \theta^*$, which means that there are too few vacancies and vice versa.

4 Endogenous Separation

This note continues the DMP series (Notes 1–4) and supplies the **canonical endogenous-separation** model: Mortensen and Pissarides (1994), as in Pissarides (2000, Ch. 2).

Notation. Idiosyncratic match productivity x is drawn on $[0,1]$; a match is hit by a Poisson shock at rate λ, upon which a new draw $x' \sim F$ (with $F(1) = 1$) replaces x.

New matches begin at the top, $x = 1$. R is the endogenous reservation productivity: a match survives a shock iff $x' \geq R$. Wages split the match surplus by generalized Nash bargaining with worker share α. r is the continuous-time discount rate, κ_v the vacancy cost, b the flow value of unemployment. Market tightness $\theta = v/u$, job-finding rate $p(\theta) = \theta q(\theta)$, vacancy-filling rate $q(\theta)$, matching elasticity $\eta(\theta) = -q'(\theta)\theta/q(\theta)$, constant under the model's Cobb–Douglas $M(u,v) = \Psi u^{\eta} v^{1-\eta}$.

Unlike Notes 1–3, this note carries **no exogenous separation** s: destruction here is purely endogenous, exactly as in MP (1994).

4.1 Environment

A firm and worker match at productivity $x = 1$. While matched, the pair draws a new $x' \sim F$ at Poisson rate λ (idiosyncratic, i.i.d. across draws); on arrival the match either continues at x' (if $x' \geq R$) or is destroyed and both parties return to the pool (if $x' < R$), **where R is determined below by mutual consent of the pair.**

Between shocks the match produces flow output x, split into wage $w(x)$ (worker) and profit $x - w(x)$ (firm) by Nash bargaining. Unemployed workers search, matching firms with open vacancies through $M(u,v)$; a filled vacancy starts at $x = 1$.

4.2 Match values

Let $J(x)$, $W(x)$, U be the values of a filled job, an employed worker, and an unemployed worker. As in Notes 1–3, write the discrete-Δ Bellman equation and take $\Delta \to 0$.

Firm, $J(x)$. Over a period of length Δ: flow profit $[x - w(x)]\Delta$; with probability $\lambda\Delta$ a shock arrives (new draw $x' \sim F$: continues at value $J(x')$ if $x' \geq R$, destroyed at value 0 if $x' < R$, so the continuation is $\int_R^1 J(s)dF(s)$); with probability $1-\lambda\Delta$ no shock, value $J(x)$:

$$J(x) = [x - w(x)]\Delta + \frac{1}{1+r\Delta}\left[\lambda\Delta\int_R^1 J(s)dF(s) + (1-\lambda\Delta)J(x)\right].$$

Multiply by $(1+r\Delta)$, collect the $J(x)$ terms on the left ($(1+r\Delta)-(1-\lambda\Delta) = (r+\lambda)\Delta$), divide by Δ, and let $\Delta \to 0$:

$$\begin{aligned}
J(x) &= [x - w(x)]\Delta + \frac{1}{1+r\Delta}\left[\lambda\Delta\int_R^1 J(s)dF(s) + (1-\lambda\Delta)J(x)\right] \\
(1+r\Delta)J(x) &= (1+r\Delta)[x - w(x)]\Delta + \lambda\Delta\int_R^1 J(s)dF(s) + (1-\lambda\Delta)J(x) \\
r\Delta J(x) &= (1+r\Delta)[x - w(x)]\Delta + \lambda\Delta\int_R^1 J(s)dF(s) - \lambda\Delta J(x) \\
(r+\lambda)\,\Delta J(x) &= (1+r\Delta)[x - w(x)]\Delta + \lambda\Delta\int_R^1 J(s)dF(s) \\
(r+\lambda)\,J(x) &= (1+r\Delta)[x - w(x)] + \lambda\int_R^1 J(s)dF(s)
\end{aligned}$$

by taking $\Delta \to 0$:

$$(r+\lambda)\,J(x) = x - w(x) + \lambda\int_R^1 J(s)dF(s)$$

or

$$rJ(x) = x - w(x) + \lambda\int_R^1 J(s)dF(s) - \lambda J(x)$$

Worker, $W(x)$. Identical steps; the only difference is that a bad draw ($x' < R$) sends the worker to unemployment (value U), not to 0:

$$
\begin{aligned}
W(x) &= w(x)\Delta + \frac{1}{1+r\Delta}\left\{\lambda\Delta\left[\int_R^1 W(s)\,dF(s) + F(R)\,U\right] + (1-\lambda\Delta)\,W(x)\right\} \\
(1+r\Delta)\,W(x) &= (1+r\Delta)\,w(x)\Delta + \lambda\Delta\left[\int_R^1 W(s)\,dF(s) + F(R)\,U\right] + (1-\lambda\Delta)\,W(x) \\
(r+\lambda)\,W(x) &= (1+r\Delta)\,w(x) + \lambda\left[\int_R^1 W(s)\,dF(s) + F(R)\,U\right] \\
&= w(x) + \lambda\left[\int_R^1 W(s)\,dF(s) + F(R)\,U\right] \\
rW(x) &= w(x) + \lambda\left[\int_R^1 W(s)\,dF(s) + F(R)\,U - W(x)\right]
\end{aligned}
$$

Unemployed, U. Flow value b; matches at rate $p(\theta)$ into a new job at $x = 1$:

$$
\begin{aligned}
U &= b\Delta + \frac{1}{1+r\Delta}\left[p(\theta)\Delta W(1) + (1-p(\theta)\Delta)U\right] \\
(1+r\Delta)\,U &= (1+r\Delta)\,b\Delta + p(\theta)\Delta W(1) + (1-p(\theta)\Delta)U \\
\big(r+p(\theta)\big)\,U &= b + p(\theta)\,W(1) \\
rU &= b + p(\theta)\,[W(1) - U]
\end{aligned}
$$

Nash split. The wage splits the joint surplus $S(x) \equiv J(x) + W(x) - U$ (the vacancy's outside option is 0 under free entry, Section 3) so that the worker gets share α:

$$
\begin{aligned}
J(x) &= (1-\alpha)\,S(x) \\
W(x) - U &= \alpha S(x)
\end{aligned}
$$

4.3 The surplus and the job-destruction condition

Now we try to cancel the wage term $w(x)$:

$$
\begin{aligned}
rJ(x) + rW(x) &= x - w(x) + \lambda\int_R^1 J(s)dF(s) - \lambda J(x) + w(x) + \lambda\left[\int_R^1 W(s)\,dF(s) + F(R)\,U - W(x)\right] \\
r\,[J(x) + W(x)] &= x + \lambda\left\{\int_R^1 [J(s) + W(s)]\,dF(s) - J(x) + F(R)\,U - W(x)\right\}
\end{aligned}
$$

Now given $S(x) = J(x) + W(x) - U$we then try to eliminate all J, W:

$$
\begin{aligned}
r[S(x)+U] &= x + \lambda \left\{ \int_R^1 [S(s)+U]\, dF(s) - [S(x)+U] + F(R)\, U \right\} \\
rS(x) &= x - rU + \lambda \left\{ \int_R^1 S(s)\, dF(s) + U[1-F(R)] - [S(x)+U] + F(R)\, U \right\} \\
&= x - rU + \lambda \int_R^1 S(s)\, dF(s) - \lambda S(x) \\
(r+\lambda) S(x) &= x - rU + \lambda \int_R^1 S(s)\, dF(s)
\end{aligned}
$$

So the integral on the rhs doesn't depend on x, so $S(x)$ is affine and increasing in x. A match is destroyed exactly when surplus turns negative, so R is then defined by $S(R) = 0$:

$$
\begin{aligned}
(r+\lambda) S(R) = R - rU + \lambda \int_R^1 S(s)\, dF(s) &= 0 \\
0 &= R - rU + \lambda \int_R^1 S(s)\, dF(s)
\end{aligned}
$$

so

$$
\begin{aligned}
(r+\lambda) S(x) &= x - R \\
S(x) &= \frac{x-R}{r+\lambda}
\end{aligned}
$$

Given this closed form of $S(x)$, then we have:

$$
\begin{aligned}
\int_R^1 S(s)\, dF(s) &= \int_R^1 \frac{s-R}{r+\lambda} dF(s) \\
&= \frac{1}{r+\lambda} \int_R^1 (s-R)\, dF(s)
\end{aligned}
$$

then we have job destruction condition:

$$
\begin{aligned}
0 &= R - rU + \lambda \int_R^1 S(s)\, dF(s) \\
rU &= R + \frac{\lambda}{r+\lambda} \int_R^1 (s-R)\, dF(s)
\end{aligned}
$$

The $\frac{\lambda}{r+\lambda}\int_R^1 (s-R)\, dF(s)$ is an option value: continuing lets the pair keep a good future draw and discard a bad one, which pulls R below the myopic cutoff rU. Then combine

$$\begin{aligned} rU &= b + p(\theta)\,[W(1) - U] \\ &= b + p(\theta)\,\alpha S(1) \\ &= b + p(\theta)\,\alpha \frac{1-R}{r+\lambda} \end{aligned}$$

Then we substitute it into $rU = R + \frac{\lambda}{r+\lambda}\int_R^1 (s-R)\, dF(s)$:

$$\Phi_2(\theta, R) \equiv R + \frac{\lambda}{r+\lambda}\int_R^1 (s-R)\, dF(s) - \left[b + p(\theta)\,\alpha \frac{1-R}{r+\lambda} \right] = 0$$

And this is the so called job-destruction curve.

4.4 Vacancies and job-creation condition

Let V be the value of an open vacancy. It fills at rate $q(\theta)$, landing the firm at the top of the prod. $J(1)$:

$$\begin{aligned} V &= -\kappa_v \Delta + \frac{1}{1+r\Delta}\left[q(\theta)\, J(1)\,\Delta + \big(1 - q(\theta)\,\Delta\big)\, V \right] \\ (1+r\Delta)\, V &= -(1+r\Delta)\,\kappa_v \Delta + q(\theta)\, J(1)\,\Delta + \big(1 - q(\theta)\,\Delta\big)\, V \\ rV &= -(1+r\Delta)\,\kappa_v + q(\theta)\, J(1) - q(\theta)\, V \\ &= -\kappa_v + q(\theta)\, J(1) - q(\theta)\, V \end{aligned}$$

Free entry drives V to 0:

$$0 = -\kappa_v + q(\theta)\, J(1)$$

so

$$\begin{aligned} J(1) &= \frac{\kappa_v}{q(\theta)} = (1-\alpha)\, S(1) \\ &= (1-\alpha)\, \frac{1-R}{r+\lambda} \end{aligned}$$

so that we have now another curve:

$$\Phi_1(\theta, R) \equiv \frac{\kappa_v}{q(\theta)} - (1-\alpha)\frac{1-R}{r+\lambda} = 0$$

which is now job-creation curve. Reading it: the left side is the expected cost of hiring one worker (κ_v per unit time, waiting $1/q(\theta)$ periods); the right side is the firm's cut of the best possible match. Entry equates the two.

4.5 Equilibrium: existence, uniqueness and the Beveridge curve

Slopes. Differentiate implicitly. For job creation: $\partial\Phi_1/\partial\theta = -\kappa_v q'(\theta)/q(\theta)^2 > 0$ (since $q' < 0$) and $\partial\Phi_1/\partial R = (1-\alpha)/(r+\lambda) > 0$, so along $\Phi_1 = 0$,

$$R'_{\mathrm{JC}}(\theta) = -\frac{\partial\Phi_1/\partial\theta}{\partial\Phi_1/\partial R} < 0.$$

For job destruction: $\partial\Phi_2/\partial\theta = -\alpha(1-R)p'(\theta)/(r+\lambda) < 0$ (since $p' > 0$), and by Leibniz's rule

$$\frac{d}{dR}\int_R^1 (s-R)\,dF(s) = 0 - 0 + \int_R^1 -1\,dF(s) = F(R) - 1$$

so

$$\frac{\partial\Phi_2}{\partial R} = \underbrace{1 - \frac{\lambda\big(1-F(R)\big)}{r+\lambda}}_{>0,\text{since } \lambda/(r+\lambda)<1} + \underbrace{\frac{\alpha p(\theta)}{r+\lambda}}_{>0} > 0,$$

$$R'_{\mathrm{JD}}(\theta) = -\frac{\partial\Phi_2/\partial\theta}{\partial\Phi_2/\partial R} > 0.$$

So $R_{\mathrm{JC}}(\theta)$ is strictly decreasing and $R_{\mathrm{JD}}(\theta)$ strictly increasing.

Existence and uniqueness. If an eq. exists, then the uniqueness is trivial: define:

$$g(\theta) = R_{JC}(\theta) - R_{JD}(\theta)$$
$$g'(\theta) = R'_{JC}(\theta) - R'_{JD}(\theta)$$

and since $\partial\Phi_1/\partial R \neq 0, \frac{\partial\Phi_2}{\partial R} \neq 0$ everywhere, g is continuous and:

$$g'(\theta) = R'_{JC}(\theta) - R'_{JD}(\theta) < 0, \forall\theta$$

Now consider the case $R = 1$:

$$\begin{aligned}\Phi_2(\theta, 1) &= 1 + \frac{\lambda}{r+\lambda}\int_1^1 (s-1)\, dF(s) - \left[b + p(\theta)\,\alpha\frac{1-1}{r+\lambda}\right] \\ &= 1 - b > 0\end{aligned}$$

and since $\frac{\partial\Phi_2}{\partial R} > 0$, we have $R_{JD}(\theta) < 1, \forall\theta > 0.$ Given CD matching function, we have $q(\theta) = \Psi\theta^{-\eta} \to \infty$ and $p(\theta) = \Psi\theta^{1-\eta} \to 0$ as $\theta \to 0^+$.From

$$\Phi_1(\theta, R) \equiv \frac{\kappa_v}{q(\theta)} - (1-\alpha)\frac{1-R}{r+\lambda} = 0$$

we have

$$R_{JC}(\theta) = 1 - \frac{(r+\lambda)\,\kappa_v}{(1-\alpha)\,q(\theta)}$$

and

$$\begin{aligned}\lim_{\theta\to 0^+} 1 - \frac{(r+\lambda)\,\kappa_v}{(1-\alpha)\,q(\theta)} &= \lim_{\theta\to 0^+} 1 - \frac{(r+\lambda)\,\kappa_v}{(1-\alpha)\infty} \\ &= 1\end{aligned}$$

Then consider job destruction:

$$\begin{aligned}\Phi_2(0, R) &\equiv R + \frac{\lambda}{r+\lambda}\int_R^1 (s-R)\, dF(s) - \left[b + 0\times\alpha\frac{1-R}{r+\lambda}\right] \\ &= R + \frac{\lambda}{r+\lambda}\int_R^1 (s-R)\, dF(s) - b\end{aligned}$$

so $\Phi_2(0,R)$is strictly increasing in R and given

$$\begin{aligned}\lim_{R\to 0^+} R+\frac{\lambda}{r+\lambda}\int_R^1 (s-R)\,dF(s)-b &= \lim_{R\to 0^+} 0+\frac{\lambda}{r+\lambda}\int_0^1 s dF(s)-b\\ &=\frac{\lambda}{r+\lambda}\int_0^1 s dF(s)-b\\ &=\frac{\lambda}{r+\lambda}\mathbb{E}[s]-b\end{aligned}$$

By regularity condition, the value of never separating $\frac{\lambda}{r+\lambda}\mathbb{E}[s]$ should be lower than home production b,[2] so $\Phi_2(0,0)<0, \Phi_2(0,1)>0$, and by intermediate value theorem, there exist $R_0\in(0,1)$ such that $\Phi_2(0,R_0)=0$ and

$$\lim_{\theta\to 0^+} R_{JD}(\theta)=R_0$$

Given the 2 limits,

$$\begin{aligned}\lim_{\theta\to 0^+} g(\theta) &= \lim_{\theta\to 0^+} R_{JC}(\theta)-R_{JD}(\theta)\\ &=1-R_0>0\end{aligned}$$

Now we consider the case where $\theta\to\infty$:

$$\lim_{\theta\to\infty} 1-\frac{(r+\lambda)\kappa_v}{(1-\alpha)\,q(\theta)}=-\infty$$

and we already have $R_{JD}(\theta)<1, \forall\theta>0$. So directly we have

$$\lim_{\theta\to\infty} g(\theta)=\lim_{\theta\to\infty} R_{JC}(\theta)-R_{JD}(\theta)=-\infty$$

Then by intermediate value theorem, we now have

$$\exists\theta^*\in(0,\infty)\,, s.t. g\left(\theta^*\right)=0$$

[2] If b fell below it, unemployment would be so unattractive that the pair would accept literally any productivity, including arbitrarily bad ones. R would be pinned at the corner 0 and no endogenous separation would ever occur, defeating the point of the model.

and therefore letting $R^* = R_{JC}(\theta^*) = R_{JD}(\theta^*)$, we have this unique eq.

$$\left(\theta^*, R^*\right)$$

Beveridge curve. Separation is no longer an exogenous constant: a match ends when a shock (rate λ) draws below R (probability $F(R)$), so the endogenous separation *rate* is $\lambda F(R)$. Steady state equates flows in and out of unemployment:

$$\underbrace{(1-u)\lambda F(R)}_{\text{employed}\to\text{unemployed}} = \underbrace{u p(\theta)}_{\text{unemployed}\to\text{employed}} \tag{6}$$

so

$$u = \frac{\lambda F(R)}{\lambda F(R) + p(\theta)}$$

Evaluated at (θ^*, R^*) this gives u^*; in (u, v) space ($v = \theta u$) it is a downward-sloping Beveridge curve, exactly as in the exogenous-separation baseline, except the separation end is now endogenous and shifts with any parameter that moves R^*.