Two-Period Brock–Mirman Model

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2026-03-10

Table of contents

1 Two-Period Brock–Mirman Model

We consider a two-period Brock–Mirman model with general period utility $u(c)$, rather than imposing a specific functional form.

1.1 Environment

Time is indexed by $t = 0, 1$. The initial capital stock $k_0 > 0$ is given. Output is produced according to the stochastic production function

$$y_t = z_t k_t^{\alpha}, \qquad 0 < \alpha < 1,$$

where z_t is an exogenous productivity shock.

We assume full depreciation of capital, so the resource constraint in each period is

$$c_t + k_{t+1} = z_t k_t^{\alpha}.$$

1.2 Planner's Problem

Using no bequest terminal condition, the agent consumes all remaining resources in period 1, so

$$k_2 = 0.$$

Hence the two-period problem is

$$\max_{\{c_0, k_1, c_1\}} \mathbb{E}_0 \left[u(c_0) + \beta u(c_1) \right]$$

subject to

$$c_0 + k_1 = z_0 k_0^\alpha,$$

$$c_1 = z_1 k_1^\alpha,$$

with feasibility conditions

$$c_0 > 0, \qquad c_1 > 0, \qquad k_1 \geq 0,$$

and given initial condition

$$k_0 \text{ is given.}$$

1.3 Reduced Form

Substituting the constraints into the objective function, the problem can be written as

$$\max_{k_1 \geq 0} \mathbb{E}_0 \left[u\big(z_0 k_0^\alpha - k_1\big) + \beta u\big(z_1 k_1^\alpha\big) \right].$$

This gives the complete formulation of the two-period Brock–Mirman model under terminal condition A while keeping the utility function in the general form $u(\cdot)$.

1.4 Solution

We solve the model by *backward substitution.*

From the period-0 resource constraint,

$$c_0 = z_0 k_0^\alpha - k_1.$$

From the period-1 resource constraint and terminal condition $k_2 = 0$,

$$c_1 = z_1 k_1^\alpha.$$

Substituting these two expressions into the objective function, the planner's problem becomes

$$\max_{k_1 \geq 0} \; \mathbb{E}_0 \left[u\big(z_0 k_0^\alpha - k_1\big) + \beta u\big(z_1 k_1^\alpha\big) \right].$$

Therefore, the only choice variable is k_1.

1.5 First-Order Condition

Assuming an interior solution, the first-order condition with respect to k_1 is

$$-u'\big(z_0 k_0^\alpha - k_1\big) + \beta \mathbb{E}_0 \left[u'\big(z_1 k_1^\alpha\big)\, z_1 \alpha k_1^{\alpha-1} \right] = 0.$$

Rearranging, we obtain

$$u'(c_0) = \beta \mathbb{E}_0 \left[u'(c_1)\, \alpha z_1 k_1^{\alpha-1} \right].$$

This is the Euler equation for the two-period problem.

1.6 Interpretation

The left-hand side, $u'(c_0)$, is the marginal utility loss from reducing current consumption by one unit in order to increase saving.

The right-hand side is the discounted expected marginal utility gain in period 1. Saving one more unit in period 0 increases k_1 by one unit, which raises next-period output and consumption by

$$\alpha z_1 k_1^{\alpha-1}.$$

Hence the optimal choice of k_1 equates the marginal cost of saving today with the discounted expected marginal benefit tomorrow.

1.7 Characterization of the Optimal Allocation

Once the optimal k_1^* is determined from

$$u'\left(z_0 k_0^\alpha - k_1\right) = \beta \mathbb{E}_0 \left[u'\left(z_1 k_1^\alpha\right) \alpha z_1 k_1^{\alpha-1}\right],$$

the optimal consumptions are given by

$$c_0^* = z_0 k_0^\alpha - k_1^*,$$

and

$$c_1^* = z_1 (k_1^*)^\alpha.$$

Therefore, solving the model amounts to solving the first-order condition for k_1^* and then recovering optimal consumption from the resource constraints.

1.8 Special Case: Log Utility

If we further assume

$$u(c) = \ln c,$$

then

$$u'(c) = \frac{1}{c}.$$

The Euler equation becomes

$$\frac{1}{z_0 k_0^\alpha - k_1} = \beta \mathbb{E}_0 \left[\frac{1}{z_1 k_1^\alpha} \alpha z_1 k_1^{\alpha-1}\right].$$

Inside the expectation, the term simplifies as

$$\frac{1}{z_1 k_1^\alpha} \alpha z_1 k_1^{\alpha-1} = \frac{\alpha}{k_1}.$$

Hence

$$\frac{1}{z_0 k_0^\alpha - k_1} = \beta \frac{\alpha}{k_1}.$$

Solving for k_1, we obtain

$$k_1^* = \frac{\alpha\beta}{1+\alpha\beta} z_0 k_0^\alpha.$$

Then optimal period-0 consumption is

$$c_0^* = z_0 k_0^\alpha - k_1^* = \frac{1}{1+\alpha\beta} z_0 k_0^\alpha,$$

and optimal period-1 consumption is

$$c_1^* = z_1 \left(\frac{\alpha\beta}{1+\alpha\beta} z_0 k_0^\alpha \right)^\alpha .$$

1.9 Conclusion

To solve the two-period Brock–Mirman model with no bequest terminal condition, we proceed in three steps:

1. Use the resource constraints to substitute out c_0 and c_1.
2. Rewrite the problem as a maximization over k_1 only.
3. Derive the first-order condition and solve for k_1^*.

Under general utility $u(\cdot)$, the solution is characterized implicitly by the Euler equation. Under log utility, the solution can be obtained in closed form.