Guess and Verify

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1 A Simple Representative Growth Model

1.1 Brock and Mirman (1972)

- The social planner solves the following problem:

$$\max_{\{c_t, k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t \log(c_t)$$
$$c_t + k_{t+1} \leq A k_t^{\alpha}$$
$$k_0 \text{ given, } c_t, k_t \geq 0 \text{ for all } t$$

- The controls are c_t and k_{t+1}, and the only state variable is k_t. We can reformulate this problem as a dynamic programming problem as follows:

$$V(k) = \max_{k'} \{\log(Ak^{\alpha} - k') + \beta V(k')\} \tag{1}$$

- If we solve for the capital stock policy rule $k' = g_k(k)$, we can obtain the policy function for consumption $c = g_c(k)$ directly from the resource constraint:

$$g_c(k) = Ak^{\alpha} - g_k(k)$$

1.2 Step 1: Guess a solution

Given

$$u(c) = \log(c)$$

Guess a solution for the value function:

$$V(k) = a + b\log(k)$$

where $a, b \in \mathbb{R}$ are constants.

1.3 Step 2: Verify the solution

Substitute the guess into the Bellman equation Equation 1:

$$a + b\log(k) = \max_{k'} \{\log\left(Ak^{\alpha} - k'\right) + \beta(a + b\log(k'))\}$$

Consider the first-order condition:

$$\frac{\partial}{\partial k'} \{\log\left(Ak^{\alpha} - k'\right) + \beta(a + b\log(k'))\} = 0$$

That is,

$$-\frac{1}{Ak^{\alpha} - k'} + \beta b\frac{1}{k'} = 0$$

We can then solve for k':

$$k' = \frac{\beta b}{\beta b + 1} Ak^{\alpha} \tag{2}$$

By the Envelope Theorem, we have:

$$V'(k) = \frac{\partial}{\partial k}\log(Ak^{\alpha} - k') = \frac{A\alpha k^{\alpha-1}}{Ak^{\alpha} - k'}$$

Given the form $V(k) = a + b\log(k)$, we have:

$$V'(k) = \frac{b}{k}$$

So,

$$\frac{A\alpha k^{\alpha-1}}{Ak^{\alpha} - k'} = \frac{b}{k}$$

We can then solve for k' again:

$$k' = (1 - \frac{\alpha}{b})Ak^{\alpha} \tag{3}$$

Combining Equation 2 and Equation 3, we obtain:

$$\frac{\beta b}{\beta b + 1} = (1 - \frac{\alpha}{b})$$

Solving for b, we get:

$$b = \frac{\alpha}{1 - \alpha\beta}$$

1.4 Step 3: Solve for the policy function

Given b and Equation 2, we can solve for the optimal capital stock k' as follows:

$$g_k(k) = k' = \alpha\beta A k^{\alpha}$$

and therefore the policy function for consumption is:

$$g_c(k) = Ak^{\alpha} - g_k(k) = (1 - \alpha\beta)Ak^{\alpha}$$

1.5 Step 4: Finalize the solution

We then solve for a by substituting $g_k(k)$ and $g_c(k)$ into the value function:

$$a + b\log(k) = \log((1 - \alpha\beta)Ak^{\alpha}) + \beta(a + b\log(\alpha\beta Ak^{\alpha}))$$

For the RHS, we have:

$$\begin{aligned}
\log((1 - \alpha\beta)Ak^{\alpha}) + \beta(a + b\log(\alpha\beta Ak^{\alpha})) &= \alpha\log(k) + \alpha\beta b\log(k) + \log((1 - \alpha\beta)A) + \beta a + \beta b\log(\alpha\beta A) \\
&= [\alpha + \alpha\beta b]\log(k) + [\beta a] + [\log((1 - \alpha\beta)A) + \beta b\log(\alpha\beta A)] \\
&= b\log(k) + [\beta a] + [\log((1 - \alpha\beta)A) + \beta b\log(\alpha\beta A)]
\end{aligned}$$

By comparing the coefficients of $\log(k)$ and the constant term, we get:

$$a = [\beta a] + [\log((1 - \alpha\beta)A) + \beta b\log(\alpha\beta A)]$$

Thus, we can solve for a:

$$a = \frac{\log((1 - \alpha\beta)A) + \beta b\log(\alpha\beta A)}{1 - \beta}$$

References

Brock, William A., and Leonard J. Mirman. 1972. "Optimal Economic Growth and Uncertainty: The Discounted Case." *Journal of Economic Theory* 4 (3): 479–513. https://doi.org/10.1016/0022-0531(72)90135-4.