Assignment 3 *

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Model Setup

Individual Problem

There is a continuum (measure one) of infinitely lived households subject to uninsured idiosyncratic labor endowment shocks and a no-borrowing constraint. Each household chooses consumption $c_t \geq 0$ and next-period asset holdings a_{t+1} to maximize

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(c_t), \qquad U(c) = \frac{c^{1-\mu} - 1}{1-\mu},$$

subject to the budget constraint

$$c_t + a_{t+1} = w l_t + (1+r) a_t,$$

and the borrowing (liquidity) constraint

$$a_{t+1} \geq 0, \qquad c_t \geq 0,$$

where r is the interest rate on assets, w is the wage rate, and l_t is the individual labor endowment shock. The logarithm of the labor endowment follows an AR(1) process:

$$\log l_t = \rho \log l_{t-1} + \sigma(1-\rho^2)^{1/2} \varepsilon_t, \qquad \varepsilon_t \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1),$$

with $\sigma \in \{0.2, 0.4\}$ (coefficient of variation of earnings) and $\rho \in \{0, 0.3, 0.6, 0.9\}$ (serial correlation). This continuous process is approximated by a seven-state Markov chain using the method of Tauchen (1986).

Aggregate Production

A representative firm uses a Cobb–Douglas technology:

$$F(K, L) = K^{\alpha} L^{1-\alpha}.$$

Since individual labor supplies are randomly inelastic and independent across agents, the per capita labor supply is constant and normalized to unity ($L = 1$). Competitive factor markets imply

$$r = \alpha K^{\alpha-1} - \delta, \qquad w = (1-\alpha) K^{\alpha},$$

where K is the aggregate (per capita) capital stock and δ is the depreciation rate.

*Replication handout based on Aiyagari, S. R. (1994), "Uninsured Idiosyncratic Risk and Aggregate Saving," *Quarterly Journal of Economics*, 109(3), 659–684. Please submit the assignment by April 28 to chengao0716@gmail.com.

Stationary Equilibrium

A stationary (recursive) competitive equilibrium consists of value and policy functions $\{V, g\}$, a wage w^*, an interest rate r^*, an aggregate capital stock K^*, and a stationary distribution $\Phi^*(a, l)$ over individual asset holdings and labor endowments such that:

(i) **Optimality**: given (r^*, w^*), the functions V and g solve the individual Bellman equation.

(ii) **Consistency**: Φ^* is the stationary distribution induced by the policy function g and the exogenous Markov chain for l.

(iii) **Market clearing**: aggregate asset supply equals aggregate capital demand,

$$K^* = \int a \, d\Phi^*(a, l) = Ea_w,$$

and therefore $K(r^*) = Ea(r^*)$, where $K(r)$ is the capital demand implied by profit maximization and $Ea(r)$ is the aggregate asset supply at interest rate r.

The equilibrium interest rate r^* is found by bisection on the interval $(-\delta, \lambda)$, where $\lambda = (1/\beta) - 1$ is the rate of time preference. By construction $r^* < \lambda$, so precautionary motives and borrowing constraints raise aggregate capital above the full-insurance level.

Parameterization

Use Aiyagari's baseline calibration:

Parameter	Symbol	Value(s)
Discount factor	β	0.96
Capital share	α	0.36
Depreciation rate	δ	0.08
Relative risk aversion	μ	$\{1, 3, 5\}$
Earnings std. dev. (c.v.)	σ	$\{0.2, 0.4\}$
Earnings serial correlation	ρ	$\{0, 0.3, 0.6, 0.9\}$
Borrowing limit	b	0 (no borrowing)
Number of Markov states		7

The full-insurance benchmark (no uncertainty) yields $r^* = 4.17\%$ and a saving rate of 23.67%. All results in Table II of Aiyagari (1994) should be compared against this benchmark.

Replication Tasks

1. **State the model.** Write down the individual's Bellman equation. Clearly define the state variables, control variables, value function, and the policy function. Explain the role of the borrowing constraint and why it binds for some agents.

2. **Discretize the labor endowment process.** Apply the Tauchen (1986) procedure to approximate the AR(1) process for $\log l_t$ with a seven-state Markov chain. Report the resulting grid and transition matrix for at least one (σ, ρ) combination. Verify that the Markov chain implies the correct unconditional variance and autocorrelation.

3. **Solve the individual problem: Value Function Iteration (VFI).** Implement VFI on a discretized asset grid. Iterate on the Bellman equation until convergence. Report your convergence criterion and the grid size you use. You may interpolate the value function between grid points if desired.

4. **Solve the individual problem: Endogenous Grid Method (EGM).** Implement Carroll's (2006) EGM as an alternative solver for the same individual problem. Starting from an Euler equation residual condition, construct the consumption policy directly on an endogenous grid. Compare the speed and accuracy of EGM relative to VFI.

5. **Compute the stationary equilibrium.** For each pair (σ, ρ, μ), find the equilibrium interest rate r^* by bisecting on $r \in (-\delta, \lambda)$ until asset demand equals asset supply. Specifically:

   - Solve the individual problem (using either VFI or EGM) for a given r, obtaining the policy function $g(a, l; r)$.
   - Simulate a long panel or iterate the transition matrix to find the stationary distribution $\Phi^*(a, l; r)$.
   - Compute aggregate assets $Ea(r) = \int a \, d\Phi^*$ and compare with capital demand $K(r)$.
   - Bisect until $|Ea(r) - K(r)| < \epsilon$ for a tight tolerance ϵ.

6. **Replicate Table II.** For $\sigma \in \{0.2, 0.4\}$, $\rho \in \{0, 0.3, 0.6, 0.9\}$, and $\mu \in \{1, 3, 5\}$, report the equilibrium net return to capital r^* (in percent) and the aggregate saving rate $\delta K^*/F(K^*, 1)$ (in percent). Arrange your results in the same format as Aiyagari's Table II, comparing your numbers with his.

7. **Discuss the results.** Use your replication to address the following questions:

   - Why does idiosyncratic risk lower r^* and raise the saving rate relative to the full-insurance benchmark?
   - How do higher risk aversion μ, higher variability σ, and higher persistence ρ each affect the precautionary saving motive, and how does this show up in r^* and the saving rate?
   - For empirically plausible values of σ and ρ, is the quantitative effect of uninsured idiosyncratic risk on aggregate saving large or small?

Replication Targets

Your replication should recover the same qualitative pattern and similar magnitudes as the published numbers. The target values from Aiyagari's Table II are:

ρ		$\sigma = 0.2$			$\sigma = 0.4$		
		$\mu = 1$	$\mu = 3$	$\mu = 5$	$\mu = 1$	$\mu = 3$	$\mu = 5$
0	r^* (%)	4.1666	4.1456	4.0858	4.0649	3.7816	3.4177
	s (%)	23.67	23.71	23.83	23.87	24.44	25.22
0.3	r^* (%)	4.1365	4.0432	3.9054	3.9554	3.4188	2.8032
	s (%)	23.73	23.91	24.19	24.09	25.22	26.66
0.6	r^* (%)	4.0912	3.8767	3.5857	3.7567	2.7835	1.8070
	s (%)	23.82	24.25	24.86	24.50	26.71	29.37
0.9	r^* (%)	3.9305	3.2903	2.5260	3.3054	1.2894	−0.3456
	s (%)	24.14	25.51	27.36	25.47	31.00	37.63

Full insurance benchmark: $r^* = 4.17\%$, saving rate $= 23.67\%$.

In your discussion, make sure to answer at least the following two questions:

- Why does the aggregate saving rate increase substantially when ρ and σ are large?
- Why does EGM typically converge faster than VFI, and when (if ever) does VFI have advantages over EGM in this setting?

Implementation Notes

- **JAX is strongly encouraged.**
- For VFI, a fine asset grid of at least 500 points is recommended. Use linear or cubic interpolation when evaluating the value function off the grid.
- For EGM, iterate on the consumption Euler equation

$$U'(c_t) = \beta(1+r)\mathbb{E}_t\left[U'(c_{t+1})\right],$$

  using the fact that the next-period consumption function is already known on the *exogenous* grid to recover today's consumption on the *endogenous* grid.
- To obtain the stationary distribution, you may either (a) simulate a large panel of agents, or (b) directly iterate the distribution on the discretized state space until convergence.
- Report wall-clock times for both VFI and EGM for at least one (σ, ρ, μ) case so that the speed comparison is concrete.

Submission Instructions

- Submit a **Jupyter notebook** (`.ipynb`) that runs from top to bottom and reproduces your main results and tables.
- Submit a short **PDF or Markdown report** that explains the model setup, solution methods (VFI and EGM), equilibrium computation, and interpretation of the results.
- The report does not need to be long, but it should clearly describe how you moved from the Aiyagari (1994) model to your final replication output, and should include a direct comparison of VFI and EGM performance.

References

Aiyagari, S. R. (1994). Uninsured idiosyncratic risk and aggregate saving. *Quarterly Journal of Economics*, 109(3), 659–684.

Carroll, C. D. (2006). The method of endogenous gridpoints for solving dynamic stochastic optimization problems. *Economics Letters*, 91(3), 312–320.

Tauchen, G. (1986). Finite state Markov-chain approximations to univariate and vector autoregressions. *Economics Letters*, 20(2), 177–181.

Huggett, M. (1993). The risk-free rate in heterogeneous-agent incomplete-insurance economies. *Journal of Economic Dynamics and Control*, 17(5–6), 953–969.