Assignment 1 *

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Exercise 1 — Brock–Mirman Model: Sequential, Recursive, and Equivalence

(PDF submission)

Model Setup

Consider the Brock–Mirman (1972) stochastic optimal growth model. Time is discrete and infinite, $t = 0, 1, 2, \ldots$ The economy is described by the following primitives:

- **Production**: Output is given by $y_t = z_t k_t^\alpha$, where $k_t > 0$ is the capital stock, $z_t > 0$ is a stochastic productivity shock, and $\alpha \in (0, 1)$.
- **Capital accumulation**: Capital depreciates fully in each period ($\delta = 1$), so the resource constraint is

$$c_t + k_{t+1} = z_t k_t^\alpha, \qquad c_t \geq 0, \quad k_{t+1} \geq 0.$$

- **Shock process**: The log of the productivity shock follows a stationary AR(1) process:

$$\ln z_{t+1} = \rho \ln z_t + \varepsilon_{t+1}, \qquad \varepsilon_{t+1} \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, \sigma^2), \quad \rho \in (-1, 1).$$

- **Preferences**: The representative household maximises expected discounted utility with logarithmic instantaneous utility $u(c) = \ln c$ and discount factor $\beta \in (0, 1)$.
- **Initial condition**: The initial state (k_0, z_0) is given.

Questions

Part 1 — Sequential Formulation

(a) Write down the household's problem in **sequential form**. That is, state the objective function and all constraints explicitly, making clear the measurability (adaptedness) requirements on the consumption and capital sequences.

*Please submit the assignment by March 24 to chengao0716@gmail.com.

(b) Form the Lagrangian for the sequential problem. Derive the **first-order conditions** with respect to c_t and k_{t+1}, and combine them to obtain the **stochastic Euler equation**:

$$\frac{1}{c_t} = \beta \mathbb{E}_t \left[\frac{\alpha z_{t+1} k_{t+1}^{\alpha-1}}{c_{t+1}} \right].$$

(c) State the **transversality condition** (TVC) associated with this problem and provide economic intuition for why it is required.

(d) Using the method of undetermined coefficients, **conjecture and verify** that the optimal policy takes the form

$$k_{t+1}^* = \phi z_t k_t^\alpha$$

for some constant $\phi \in (0,1)$. Determine the value of ϕ and write down the implied optimal consumption policy c_t^*.

Part 2 — Recursive Formulation

(a) Identify the **state variables** of this problem and explain your choice. Write down the **Bellman equation** for the value function $V(k,z)$.

(b) Define the **Bellman operator** T and verify that it satisfies Blackwell's sufficient conditions (monotonicity and discounting). Conclude that T is a contraction mapping and hence that a unique fixed point $V = TV$ exists.

(c) Solve the Bellman equation analytically by guessing a value function of the form $V(k,z) = A + B\ln k + C\ln z$ (here A, B, C are undetermined constants, unrelated to the TFP parameter A in Exercise 2). Determine the constants A, B, and C, and derive the **optimal policy function** $g(k,z)$.

Part 3 — Proof of Equivalence

(a) **(Sequential ⇒ Recursive)** Show that if $\{c_t^*, k_{t+1}^*\}_{t=0}^\infty$ is an optimal solution to the sequential problem, then the associated value function $V(k_0,z_0) = \mathbb{E}_0 \sum_{t=0}^\infty \beta^t \ln c_t^*$ satisfies the Bellman equation.
Hint: Use the principle of optimality.

(b) **(Recursive ⇒ Sequential)** Show that the plan generated by iterating the policy function $g(k,z)$ derived from the Bellman equation is optimal for the sequential problem.
Hint: Iterate the Bellman equation N times and take $N \to \infty$. Show that the continuation term $\beta^{N+1}\mathbb{E}_0[V(k_{N+1}, z_{N+1})]$ vanishes. In this model, you may prove this either directly from the closed-form value function or by verifying the appropriate no-bubbles / transversality condition along the policy-generated path.

(c) Conclude by stating the **equivalence theorem**: the sequential problem and the Bellman equation have the same value and the same set of optimal plans. Verify that the optimal policy and the optimal consumption rule you found in Parts 1 and 2 are identical.

Exercise 2 — Stochastic Growth Model with Irreversible Investment

(Jupyter notebook submission)

Model Setup

Consider the stochastic growth model with CRRA utility and a Markov productivity shock. The social planner solves:

$$\max_{\{c_t, k_{t+1}\}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma}$$

subject to

$$\begin{aligned} c_t + i_t &\le \theta_t A k_t^{\alpha}, \\ k_{t+1} &= i_t + (1-\delta)\, k_t, \\ i_t &\ge 0, \end{aligned}$$

with k_0, θ_0 given and $c_t, k_t \ge 0$ for all t. The constraint $i_t \ge 0$ means investment is **irreversible**: the capital stock cannot be liquidated, so $k_{t+1} \ge (1-\delta) k_t$ must hold at all times.

The log of the productivity shock follows an AR(1) process

$$x_t = \rho\, x_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim \mathcal{N}(0, \sigma^2), \quad \theta_t = \exp(x_t),$$

approximated by a finite Markov chain via Tauchen's method.

Use the following **baseline parameters**:

Parameter	Symbol	Value
Discount factor	β	0.9
Capital share	α	0.33
TFP level	A	1
Depreciation rate	δ	0.1
Risk aversion	γ	2
AR(1) persistence	ρ	0.95
AR(1) shock std. dev.	σ	0.00712
No. of Markov states	N	4

Questions

(a) Bellman equation. Write down the Bellman equation for the value function $V(k, \theta)$. Identify the state variables, the control variable, and the feasibility set for the control given the irreversibility constraint. In particular, show that the irreversibility constraint can be written as a lower bound on next-period capital:

$$k' \;\ge\; (1-\delta)\, k.$$

(b) Algorithm modification. Explain, in words and/or pseudocode, how the Value Function Iteration (VFI) algorithm for the reversible-investment model (as presented in the lecture notes) must be modified to incorporate the constraint $i_t \geq 0$. Specifically:

- Which step of the algorithm changes?
- How should the return function $U(c)$ be treated for (k_i, k_j) pairs that violate $k_j \geq (1-\delta)k_i$?
- Does this modification affect the contraction property of the Bellman operator? Briefly justify your answer.

(c) Implementation. Implement the modified VFI in Python. Your notebook should follow the same five-step structure used in the lecture:

(i) Specify model and algorithm parameters; compute the steady state.

(ii) Discretize the AR(1) shock with Tauchen's method (`quantecon`) and construct the capital grid $k \in [k_{\min}, k_{\max}]$.

(iii) Build the return matrix, setting $U = -10^{10}$ (or `-np.inf`) wherever $c \leq 0$ **or** $i_t < 0$.

(iv) Iterate the Bellman operator to convergence.

(v) Recover the policy functions $g_k(k,\theta)$ and $g_c(k,\theta)$.

Report the number of VFI iterations and wall-clock time to convergence.

(d) Policy function comparison. On a single figure, plot the capital policy function $g_k(k,\theta)$ for the **lowest** and **highest** productivity states under both the reversible and irreversible specifications. Also add the line $k' = (1-\delta)k$ to indicate where the irreversibility constraint binds. Describe and interpret the differences you observe.

(e) Simulation and moments. Starting from the state (k_0, θ_0) where $k_0 = k_{ss}$ and θ_0 is the Markov state closest to $\mathbb{E}[\theta]$, simulate the irreversible model for $T = 2{,}000$ periods with a burn-in of 200. Plot time series of output y_t, consumption c_t, investment i_t, and capital k_t. Compute and report the following moments (post burn-in) for both the reversible and irreversible models:

Statistic	Reversible	Irreversible
$\mathbb{E}[y]$		
$\mathbb{E}[c]$		
$\mathbb{E}[i]$		
$\mathbb{E}[k]$		
$\text{std}(y)/\mathbb{E}[y]$		
$\text{std}(c)/\mathbb{E}[c]$		
$\text{std}(i)/\mathbb{E}[i]$		

Discuss how the irreversibility constraint affects the level and volatility of each variable, and provide economic intuition for the differences.

Submission Instructions

- **Exercise 1**: submit a single PDF file containing your written answers. All mathematical steps must be shown clearly. Statements without justification will not receive full credit. Clearly label each part (1a, 1b, ..., 3c).
- **Exercise 2**: submit a self-contained **Jupyter notebook** (`.ipynb`) that runs from top to bottom without errors and reproduces all figures and tables. Include brief written explanations in Markdown cells where indicated.

Reference

Brock, W. A. and Mirman, L. J. (1972). Optimal economic growth and uncertainty: The discounted case. *Journal of Economic Theory*, 4(3):479–513.

Stokey, N. L., Lucas, R. E., and Prescott, E. C. (1989). *Recursive Methods in Economic Dynamics*. Harvard University Press.